

ADDIS ABABA UNIVERSITY
DEPARTMENT OF MATHEMATICS
Applied Mathematics I (Math 1041) AY 2022/2023

Worksheet 3

1. Using $\varepsilon - \delta$ definition of a limit, show the following

a) $\lim_{x \rightarrow -2} (7 - 2x) = 11$ b) $\lim_{x \rightarrow 5} (x^2 - 3x) = 10$ c) $\lim_{x \rightarrow 1} \left(\frac{x+1}{x+2} \right) = \frac{2}{3}$ d) $\lim_{x \rightarrow 0} \sin 3x = 0$

2. If it exists, evaluate following limits

a) $\lim_{h \rightarrow 0} h^2 \left(1 + \frac{1}{h^2} \right)$ b) $\lim_{y \rightarrow 1} \left(\frac{y^3 - 1}{\sqrt{y} - 1} \right)$ c) $\lim_{x \rightarrow 8} \frac{\sqrt{7 + \sqrt[3]{x}} - 3}{x - 8}$

d) $\lim_{y \rightarrow 8} \left(\frac{y^{\frac{1}{3}} - 2}{y - 8} \right)$ e) $\lim_{x \rightarrow -2} \left(\frac{x^2 - 4}{x^3 - 8} \right)$ f) $\lim_{x \rightarrow 0} \frac{x^2}{\sec x - 1}$

g) $\lim_{x \rightarrow 0} \frac{2x^2 + x}{\sin x}$ h) $\lim_{x \rightarrow 0} x^2 (1 + \cot^2 3x)$ i) $\lim_{x \rightarrow \pi} \left(\frac{\sin^3 x + \sin x \cos x^2}{x - \pi} \right)$

j) $\lim_{x \rightarrow \infty} \left(\frac{2 + \sin^2 x}{x^2} \right)$ k) $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{3x} \right)^{2x+1}$ l) $\lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}}$

m) $\lim_{x \rightarrow -\infty} \left(\frac{3x + 4}{\sqrt{2x^2 - 5}} \right)$ n) $\lim_{t \rightarrow \infty} (\sqrt{t^2 + t} - \sqrt{t^2 + 4})$ o) $\lim_{x \rightarrow \infty} \frac{(x^3 + 1)^{\frac{1}{3}}}{\sqrt{x^2 - 3}}$

p) $\lim_{x \rightarrow \infty} \frac{3^x - 5^x}{3^x + 5^x}$ q) $\lim_{x \rightarrow -1} \left(\frac{|x+2| - 1}{1 - |x|} \right)$ r) $\lim_{x \rightarrow 0} \frac{|x|^3 - x^2}{x^3 + x^2}$

3. Using squeezing theorem evaluate

a) $\lim_{x \rightarrow \pi} (x - \pi) \cos^2 \left(\frac{1}{x - \pi} \right)$ b) $\lim_{x \rightarrow 0} \sin x \cos \frac{1}{x}$ c) $\lim_{x \rightarrow 0} x^4 \sin \frac{1}{\sqrt[5]{x^3}}$

d) $\lim_{x \rightarrow -2} g(x)$, if $|g(x) - 3| < 5(x+2)^2 \forall x$ e) $\lim_{x \rightarrow -\frac{\pi}{2}} f(x)$, if $4 - 2 \sin x \leq 2 f(x) + 4 \leq 8 + 2 \sin x \forall x \in (-\pi, 0)$

4. Find all vertical asymptotes of graph of f.

a) $f(x) = \frac{x^2 - 5x + 6}{x^3 - 8}$ b) $f(x) = \frac{8x - 2x^2}{x^2 - 16}$ c) $f(x) = \frac{\sin(x^2 - 1)}{x^3 - x}$

5. Find all horizontal asymptotes of graph of f.

a) $f(x) = \frac{x^2 - 5x + 6}{x^3 - 8}$ b) $f(x) = \frac{(1 - x^2)(x+1)}{x^2(1 - 2x)}$ c) $f(x) = \frac{x}{\sqrt{x^2 + 1}}$

6. Show each of the following.

a) $\lim_{x \rightarrow a} f(x) = 0$ if and only if $\lim_{x \rightarrow a} |f(x)| = 0$. B) If $\lim_{x \rightarrow a} f(x) = L$, then $\lim_{x \rightarrow a} |f(x)| = |L|$.

7. a) Suppose $\lim_{x \rightarrow a} f(x) = L \in \mathbb{R}$ and $\lim_{x \rightarrow a} (f(x)g(x)) = 1$. What can we say about $\lim_{x \rightarrow a} g(x)$?

b) If $\lim_{x \rightarrow a} (f(x) + g(x))$ and $\lim_{x \rightarrow a} f(x)$ exist, what can we say about $\lim_{x \rightarrow a} g(x)$?

8. Prove that if there is a number M such that $\left| \frac{f(x) - L}{x - a} \right| \leq M, \forall x \neq a$, then $\lim_{x \rightarrow a} f(x) = L$.
9. If $\lim_{x \rightarrow a} f(x) = 0$ and $|g(x)| \leq M \forall x \neq a$, then show that $\lim_{x \rightarrow a} (fg)(x) = 0$.
10. Let $f(x) = \begin{cases} ax - b & x \leq 1 \\ 3x & 1 < x < 2 \\ bx^2 - a & 2 \leq x \end{cases}$. Find numbers a and b such that f is continuous on $\mathbb{R}$.
11. Find the value of k so that $f(x) = \begin{cases} \frac{\cos x - 1}{\sin 2x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at 0 .
12. Determine for which of the following functions we can define $f(a)$ so as to make f continuous at a .
- a) $f(x) = \frac{\tan 3x}{2x^2 - 5x}; a = 0$ b) $f(x) = \frac{x^2 - 1}{|x - 1|}; a = 1$ c) $f(x) = \begin{cases} x^2 + 1 & \text{for } x < -1 \\ x & \text{for } x > -1 \end{cases}; a = -1$
13. Using the intermediate value theorem show that the equation has a root in the given interval.
- a) $2x^3 - 4x^2 + 5x = 4, [1, 2]$ b) $2\cos x = x^2, [0, \frac{\pi}{2}]$ c) $\tan x = 1 - x, [0, \frac{\pi}{4}]$
14. Show that if f is continuous and $0 \leq f(x) \leq 1 \forall x \in [0, 1]$, then there exists at least one number c in $[0, 1]$ for which $f(c) = c$. (**Hint** apply IVT on $g(x) = x - f(x)$)
15. Find the set of all points where the derivative exists for
- a. $f(x) = |\sin x|$ b. $f(x) = [x]$ c. $f(x) = \begin{cases} \sqrt{x} & \text{for } 0 \leq x \leq 1 \\ x^2 & \text{else where} \end{cases}$ d. $f(x) = \begin{cases} x \cos(\frac{1}{x}), & x \neq 0 \\ 0 & , x = 0 \end{cases}$
16. If f and g are two functions with $f(a) = 1, f'(a) = 2, g(a) = -1$ and $g'(a) = 3$, then find $(fg + f)'(a)$ and $\left(\frac{f}{f-g}\right)'(a)$
17. Let f be a differentiable function such that $f'(x) = e^x$, then find $\frac{d}{dx}(f(\ln(\ln 2x)))$
18. Let $f(x) = x^2|x|$. Show that $f''(x)$ is continuous at 0 but not differentiable at 0 .
19. Discuss the continuity and differentiability of each function at the given point a
- a. $f(x) = \sqrt[3]{x} - x$ at $a = 0$ b. $f(x) = \begin{cases} \frac{\sqrt{x}-1}{x-1} & \text{for } x \neq 1 \\ 1/2 & \text{for } x = 1 \end{cases}$ at $a = 1$ c. $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $a = 0$
20. Show that the derivative of even function is odd and vice versa.
21. Assuming $f'(a)$ exists express the following in terms of $f'(a)$
- a. $\lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{h}$ b. $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a-h)}{h}$ c. $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}, a > 0$
22. Find $f'(x)$ at each point where the derivative exists.
- a. $f(x) = \frac{-x}{(x-1)^2}$ b. $f(x) = xe^{-\ln x}$ c. $f(x) = x + |x|$
- d. $f(x) = (x-7)|x-7|$ e. $f(x) = \frac{e^x}{\sqrt{3^{2x} - 1}}$ f. $f(x) = \log_{x^2}(x^2 e^x)$

g. $f(x) = (\ln x)^x$

h. $f(x) = 3^{2x} \log_3(x^2 + 3)$

i. $f(x) = x^{x^2}$

23. Given $f(x) = \begin{cases} \cos x & x \geq 0 \\ ax + b & x < 0 \end{cases}$. Find the values of a and b such that $f'(0)$ exists

24. Given $f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax^2 + bx + c & \text{if } x \geq 1 \end{cases}$. Find the values of a and b such that $f''(1)$ exists.

25. Find the 3rd derivative of the following functions.

a. $f(x) = x^2 e^{-3x}$

b. $f(x) = \ln(\ln(5x))$

c. $f(x) = \cos^2 x - e^{\tan x}$

26. Explain why the Mean Value Theorem (MVT) does not apply for $f(x) = x^{2/3}$ on $[-1, 1]$

27. Using the MVT show that

a. if $|f'(x)| \leq 1 \forall x$ in some interval I, then $|f(x_1) - f(x_2)| \leq |x_1 - x_2| \forall x_1, x_2 \in I$

b. $|\sin x| \leq |x| \forall x \in \mathbb{R}$.

28. Find the asymptote(s), intervals of monotonicity, critical points, the local extreme points, intervals of concavity and inflection point(s) of the following functions.

a) $f(x) = -3x^4 + 4x^3$

b) $f(x) = |x^2 + x - 2|$

c) $f(x) = (x - 2)^{2/3}$

d) $f(x) = x^2 + \frac{2}{x}$

e) $f(x) = \frac{x^2}{x-2}$

f) $f(x) = \frac{x^2 - 6x}{(x+1)^2}$

g) $f(x) = x^2 e^{-x}$

h) $f(x) = |x-2| + |x-4|$

i) $f(x) = x\sqrt{1-x}$

29. Evaluate each of the following integrals.

a. $\int \frac{dx}{x(1 - \ln x)}$

b. $\int x^2 e^x dx$

c. $\int \frac{x^2 + 4}{x^3 - 2x^2 + x} dx$

d. $\int \frac{x}{\sqrt{3-2x-x^2}} dx$

e. $\int \sin^3 x dx$

f. $\int e^{-2x} \sin 3x dx$

g. $\int \sin 4x \cos 3x dx$

h. $\int \frac{dx}{\sqrt{9-4x^2}}$

i. $\int \cos x \ln(\sin x) dx$

j. $\int \frac{dx}{(x^2-1)^2}$

k. $\int \frac{x+1}{x^2(x-1)} dx$

l. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx \int \sin^3 x \cos^2 x dx$

m. $\int \sqrt{9-4x^2} dx$

n. $\int e^x \sin x dx$

o. $\int \frac{1}{\sqrt{4-x^2}} dx$

p. $\int \frac{1}{\sqrt{3-x^2-2x}} dx$

q. $\int \frac{1}{\sqrt{x}(1-x)} dx$

r. $\int \frac{\cos x}{1+\sin^2 x} dx$

s. $\int \tan^{-1} x dx$

t. $\int \ln x dx$