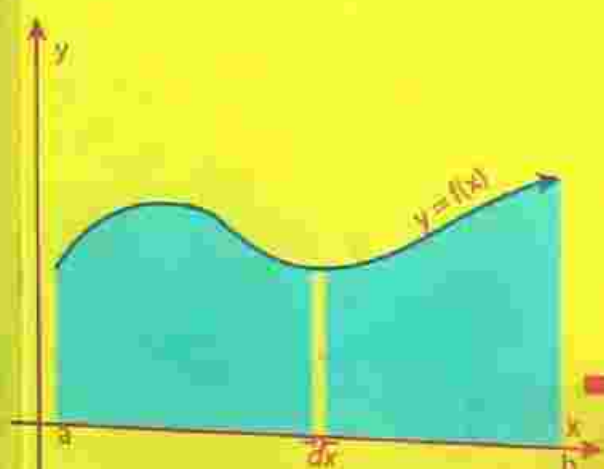


Extreme Series

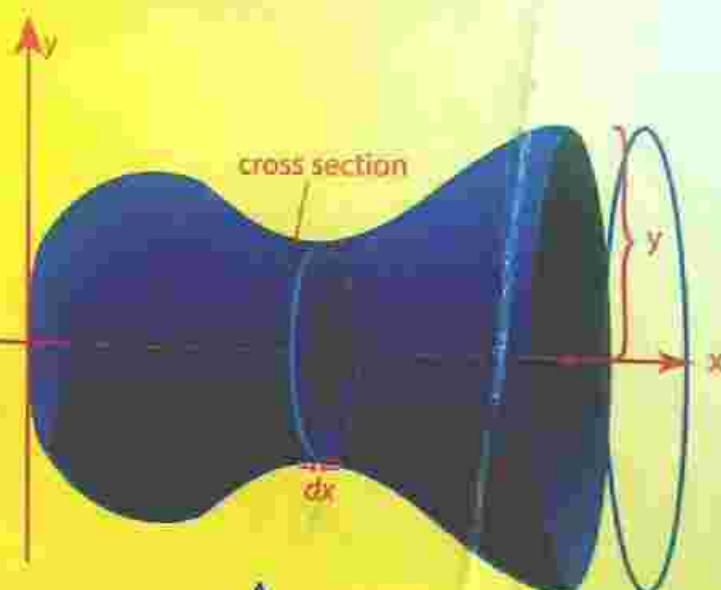
MATHEMATICS

GRADE

12



A region to be rotated



Hemispherical bowl



Based on Common Currently Used
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Sequence and Series

Sequence

Intuitively, a sequence is:

- List of things (number) in rule
- Pairing between positive integer and real number
- Special type of function whose domain is the set of positive integer. (All natural $\mathbb{N}$)

Number sequence

Definition: i) A sequence is a function with domain natural number

ii) A sequence that has a last term is called **finite sequence**. A sequence that does not have a last term is called **infinite sequence**.

- $1, 3, 5, 7, 9 \leftarrow$ finite sequence
- $1, 3, 5, 7, 9, \dots \leftarrow$ infinite sequence
- Domain of finite sequence is $\{1, 2, 3, \dots, n\}$
- Domain of infinite sequence is $\mathbb{N}$

Notation of sequence

Let function $a(n) = 2n - 1$, domain of a is the set of natural number.

Then $a(1) = 1, a(2) = 3, a(3) = 5, a(4) = 7$

- Rather than use function notation for sequence use a subscript notation $\{a_n\}, n = 1, 2, 3, \dots$

$$\Rightarrow a_1 = 1, a_2 = 3, a_3 = 5, a_4 = 7 \text{ and } a_n = 2n - 1$$

- Elements in the range of sequence $\{a_n\}$ are called **the term of a sequence**.

- The terms are $1, 3, 5, 7, \dots, 2n - 1, \dots$

- The first term is denoted by $a_1 = 1$

- The 2nd term is denoted by $a_2 = 3$

- The 3rd term is denoted by $a_3 = 5$

- The n^{th} term is denoted by $a_n = 2n - 1$

- a_n is called the general term or the n^{th} term of the sequence.

- a_n generate all the terms of sequence

- The first three terms denoted by: a_1, a_2 , and a_3

- The first five terms denoted by: a_1, a_2, a_3, a_4 and a_5

- The first n^{th} terms denoted by: $a_1, a_2, a_3, \dots, a_n$

- The sequence $\{a_1, a_2, a_3, \dots, a_n, \dots\}$ is denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Sequence can be described by:

- listing the term
- writing the general term.
- drawing graphs
- using recurrence relation.

Illustrative Example

1. Find the first four terms, 8th term and 10th term of sequence

a) $a_n = \{1 + (-1)^{n+1}\}$ c) $a_n = \{6\}$

b) $a_n = \left\{ \frac{n+2}{3n-8} \right\}$ d) $a_n = \{\log n\}$

Solution: a) $a_n = \{1 + (-1)^{n+1}\}$

$\Rightarrow a_1 = 1 + (-1)^2 = 2, a_2 = 1 + (-1)^3 = 0, a_3 = 1 + (-1)^4 = 2, a_4 = 1 + (-1)^5 = 0$

$\therefore$ The first four terms are $\{2, 0, 2, 0\}$

$\Rightarrow a_8 = 1 + (-1)^9 = 1 - 1 = 0 \Rightarrow a_{10} = 1 + (-1)^{11} = 1 - 1 = 0$

b) $a_n = \left\{ \frac{n+2}{3n-8} \right\} \Rightarrow a_1 = \frac{3}{-5}, a_2 = \frac{4}{-2} = -2, a_3 = \frac{5}{1} = 5$

$a_4 = \frac{6}{4} = \frac{3}{2}$

$\Rightarrow$ The first four terms are $\left\{ \frac{-3}{5}, -2, 5, \frac{3}{2} \right\}$

$\Rightarrow a_8 = \frac{10}{16} = \frac{5}{8}$ and $a_{10} = \frac{12}{22} = \frac{6}{11}$

c) $a_n = \{6\} \Rightarrow a_1 = a_2 = a_3 = a_4 = 6 \Rightarrow a_8 = a_{10} = 6$

d) $a_n = \{\log n\}$

$\Rightarrow a_1 = \log^1, a_2 = \log^2, a_3 = \log^3, a_4 = \log^4$

$\Rightarrow a_8 = \log^8$ and $a_{10} = \log^{10} = 1$

- A sequence can be described by recurrence relation.

- A recursively defined sequence is one in which each succeeding term of the sequence is defined by using some of the preceding terms and together with several terms.

The **Fibonic sequence** the fibonic sequence is defined recursively by:

$$a_1 = 1, a_2 = 1, a_k = a_{k-1} + a_{k-2} \text{ for } k \geq 3$$

$$\Rightarrow a_3 = a_2 + a_1 = 1 + 1 = 2$$

$$\Rightarrow a_4 = a_3 + a_2 = 2 + 1 = 3 \Rightarrow a_5 = a_4 + a_3 = 3 + 2 = 5$$

This recursive sequence 1, 1, 2, 3, 5, 8, ... is called the **Fabonic sequence**. (The Fibonic sequence is extended by adding two consecutive terms to get the next term)

Find the first five terms of sequence defined recursively

a) $a_1 = 1$ and $a_n = na_{n-1}, n \geq 2$.

b) $a_1 = 1, a_2 = 3$, and $a_n = \frac{1}{2}(a_{n-1} + a_{n-2}), n \geq 3$

Solution: a) $a_1 = 1 \Rightarrow a_2 = 2a_1 = 2(1) = 2 = 2!$

$$\Rightarrow a_3 = 3a_2 = 3(2) = 6 = 3!$$

$$\Rightarrow a_4 = 4a_3 = 4(6) = 24 = 4!$$

$$\Rightarrow a_5 = 5a_4 = 5(24) = 120 = 5!$$

- n factorial can defined recursively

b) $a_3 = \frac{1}{2}(a_2 + a_1) = \frac{1}{2}(3 + 1) = 2$

$$a_4 = \frac{1}{2}(a_3 + a_2) = \frac{1}{2}(2 + 3) = \frac{5}{2}$$

$$a_5 = \frac{1}{2}(a_4 + a_3) = \frac{1}{2}\left(\frac{5}{2} + 2\right) = \frac{9}{4}$$

Which the following define sequence or not

a) $a_n = n^2 + 1$

b) $a_n = a_{n-1} + 1$ for $n \geq 1, a_1 = 1$

c) $a_n^2 = 3n + 1$

d) $a_1 = \pm 1, a_2 = 3$, and $a_n = (a_{n-1})(a_{n-2}), n \geq 3$

Solution: Sequence is special type of function with domain natural number

a) $a_n = n^2 + 1$ is function $\Rightarrow$ sequence

b) $a_n = a_{n-1} + 1$ is function $\Rightarrow$ recursively define sequence

$$c) \quad a_n^2 = 3n + 1 \Rightarrow a_n = \pm \sqrt{3n + 1} \Rightarrow a_1 = \pm 2 \Rightarrow (1, 2), (1, -2)$$

Not function $\Rightarrow$ not sequence

$$d) \quad a_3 = (a_2)(a_1) = (3)(\pm 1) = \pm 3 \Rightarrow (3, 3), (3, -3)$$

There are two standard sequence **Arithmetic** and **geometric sequence**

Arithmetic sequence (Arithmetic progression)

- Characterized by a common difference between successive terms.
A sequence with a constant difference between consecutive terms is called an **arithmetic sequence (arithmetic progression)**

$$\text{i.e. } A_2 - A_1 = A_3 - A_2 = \dots A_n - A_{n-1} = d, \text{ for all } n \geq 1$$

- The difference between consecutive term is called the **common difference (d)**.

$$\Rightarrow A_2 = A_1 + d, A_3 = A_2 + d, \dots A_{n+1} = A_n + d$$

$$\Rightarrow A_2 = A_1 + d, A_3 = A_1 + 2d, A_4 = A_1 + 3d, \dots$$

$$\Rightarrow A_n = A_1 + (n-1)d$$

4. Which of the following is an arithmetic sequence

$$a) \quad 3, 7, 11, 15, \dots \quad c) \quad \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$$

$$b) \quad \frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, \dots \quad d) \quad 1, 4+2p, 7+4p, 10+6p, \dots$$

$$e) \quad 1, 4, 9, 16, \dots$$

Solution:

$$a) \quad 7 - 3 = 11 - 7 = 15 - 11 = 4 \Rightarrow \text{A.P. with } d = 4$$

$$b) \quad 1 - \frac{1}{3} = \frac{5}{3} - 1 = \frac{7}{3} - \frac{5}{3} = \frac{2}{3} \Rightarrow \text{A.P. with } d = \frac{2}{3}$$

$$c) \quad \frac{1}{3} - \frac{1}{2} \neq \frac{1}{4} - \frac{1}{3} \Rightarrow \text{not A.P.}$$

$$d) \quad 4+2p - 1 = 7+4p - (4+2p) = 3+2p \Rightarrow \text{A.P. with } d = 3+2p$$

5. State whether each of the following sequence is an arithmetic sequence. If it is state the common difference (d).

$$a) \quad a_n = 5 - 2n$$

$$c) \quad a_n = n^2$$

$$e) \quad a_n = 7$$

$$b) \quad a_n = 4n + 1$$

$$d) \quad a_n = \frac{5+3n}{2}$$

$$f) \quad a_n = (-1)^n$$

Solution: a) $a_{n+1} = 5 - 2(n+1) = 3 - 2n$

$$\Rightarrow a_{n+1} - a_n = 3 - 2n - (5 - 2n) = -2 \Rightarrow d = -2 \Rightarrow \text{A.P.}$$

$$3n+1 - a_n = 4(n+1)+1 - (4n+1) = 4 \Rightarrow d=4 \Rightarrow \text{A.P.}$$

$$3n+1 - a_n = (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n+1 \Rightarrow \text{not constant}$$

$$\Rightarrow \text{not arithmetic sequence}$$

$$d) a_{n+1} - a_n = \frac{5+3(n+1)}{2} - \left(\frac{5+3n}{2} \right) = \frac{5+3n+3-5-3n}{2} = \frac{3}{2}$$

$$d = \frac{3}{2} \Rightarrow \text{A.P.}$$

$$e) a_{n+1} - a_n = 7 - 7 = 0 \Rightarrow d = 0 \Rightarrow \text{A.P.}$$

$$f) -1, 1, -1, 1, \dots$$

$$\Rightarrow 1 - (-1) \neq -1 - 1 \Rightarrow 2 \neq -2 \Rightarrow \text{not A.P.}$$

If A_n is n^{th} terms of arithmetic sequence with first term A_1 and common difference d is given by $A_n = A_1 + (n-1)d$, for all $n \geq 1$

Find 16^{th} , 25^{th} and n^{th} terms of sequence $-4, 1, 6, \dots$

$$\text{Solution: } A_1 = -4, d = 1 - (-4) = 5, A_n = A_1 + (n-1)d$$

$$\Rightarrow A_{16} = -4 + (16-1)5 = -4 + 75 = 71$$

$$\Rightarrow A_{25} = -4 + (25-1)5 = -4 + 120 = 116$$

$$\Rightarrow A_n = -4 + (n-1)5 = -4 + 5n - 5 = 5n - 9$$

The 5^{th} term of an A.P. is 19 and the 11^{th} term is 43

Find a) d b) A_1 c) A_{20} d) n^{th} term

$$\text{Solution: } A_5 = A_1 + 4d = 19$$

$$A_{11} = A_1 + 10d = 43$$

$$a) \Rightarrow A_5 - A_{11} = -6d = -24 \Rightarrow d = 4$$

$$b) \text{ To find } A_1 \text{ from } A_5 = A_1 + 4d = 19 \Rightarrow A_1 + 4d = 19$$

$$\Rightarrow A_1 = 19 - 4(4) = 19 - 16 = 3$$

$$c) A_{20} = A_1 + 19d = 3 + 19(4) = 79$$

$$d) n^{\text{th}} \text{ term, } A_n = A_1 + (n-1)d \Rightarrow A_n = 3 + (n-1)4 = 4n - 1$$

If $A_7 = 27$, $d = 5$, then find a) A_{20} b) n^{th} term

$$\text{Solution: } A_7 = A_1 + 6d \Rightarrow A_1 + 6(5) = 27 \Rightarrow A_1 = -3$$

$$a) A_{20} = -3 + 19(5) = 92$$

$$b) A_n = A_1 + (n-1)d \Rightarrow A_n = -3 + (n-1)5 = 5n - 8$$

Which term of the sequence $6, 14, 22, \dots$ is

$$a) 246 \quad b) 258$$

$$\text{Solution: } a) A_1 = 6, d = 14 - 6 = 8 \Rightarrow A_n = A_1 + (n-1)d$$

$$\Rightarrow 246 = 6 + (n-1)8 \Rightarrow 8n - 2 = 246 \Rightarrow 8n = 248 \Rightarrow n = 31$$

$$\Rightarrow a_{31} = 246$$

$(31)^{\text{th}}$ term is 246

$$27 = A_1 + 5d - 5$$

$$\begin{array}{r} 27 \\ - 25 \\ \hline 2 \end{array}$$

$$b) \quad A_n = A_1 + (n-1)d \Rightarrow 6 + (n-1)8 = 258$$

$$\Rightarrow 8n - 2 = 258 \Rightarrow 8n = 260 \Rightarrow n = \frac{260}{8} = \frac{65}{2} \notin \mathbb{N}$$

$\therefore 258$ is not part of sequence

10. The p^{th} term of an A.P is q and the q^{th} term is p . Find $(p+q)^{\text{th}}$ term.

Solution: $A_n = A_1 + (n-1)d$

$$\Rightarrow \begin{cases} A_p = A_1 + (p-1)d = q \\ A_q = A_1 + (q-1)d = p \end{cases}$$

$$\Rightarrow A_p - A_q = d(p-1) - d(q-1) = d(p-1-q+1) = d(p-q)$$

$$\Rightarrow q - p = d(p-q) \Rightarrow d = \frac{p-q}{q-p} = -1$$

$$\Rightarrow A_q = A_1 + (q-1)(-1) = p \Rightarrow A_1 = p + q - 1$$

$$\Rightarrow A_{(p+q)} = A_1 + (p+q-1)d = p+q-1 + (p+q-1)(-1)$$

$$\Rightarrow A_{(p+q)} = p+q-1-p-q+1 = 0$$

11. Find the total number of integer between 100 and 1000 divisible by 7

Solution: $A_1 = 105, d = 7$

Also integer less than 1000 divisible by 7 is 994.

$$\Rightarrow \{105, 112, 119, 126, 133, \dots, 994\}$$

$$\Rightarrow A_n = 994 \Rightarrow A_n = A_1 + (n-1)d$$

$$\Rightarrow 994 = 105 + (n-1)7 \Rightarrow 7n = 994 - 105 + 7 = 896$$

$$\Rightarrow 7n = 896 \Rightarrow n = \frac{896}{7} = 128 \Rightarrow \text{the total number of integer is } 128$$

12. If the n^{th} term of an arithmetic sequence is x and its $(2n)^{\text{th}}$ term is y . Find the $(3n)^{\text{th}}$ term in terms of x and y .

Solution: $A_n = A_1 + (n-1)d$

$$A_{2n} = A_1 + (2n-1)d$$

$$A_n - A_{2n} = (n-1)d - (2n-1)d$$

$$\Rightarrow x - y = d(n-1-2n+1) = -nd \Rightarrow nd = y - x$$

$$\text{But } A_n = A_1 + (n-1)d = A_1 + nd - d$$

$$\Rightarrow x = A_1 + (y-x) - d \Rightarrow A_1 - d = 2x - y$$

$$\text{And } A_{3n} = A_1 + (3n-1)d = A_1 + 3nd - d = A_1 - d + 3nd$$

$$\Rightarrow A_{3n} = 2x - y + 3(y-x) = -x + 2y$$

46. The 7th term of an A.P is 23, the n^{th} term is 43 and $(2n)^{\text{th}}$ term is 91.

Find:

a) common difference b) A_1 c) n

Solution: $A_{2n} = A_1 + (2n - 1)d$ and $A_n = A_1 + (n - 1)d = 43$

$$\begin{aligned} \text{a) } A_{2n} - A_n &= d(2n - 1 - (n - 1)) = nd \\ \Rightarrow 91 - 43 &= nd \Rightarrow nd = 48 \end{aligned}$$

$$\text{Also, } A_7 - A_1 = nd - d - 6d = nd - 7d$$

$$\Rightarrow A_n - A_7 = nd - d - 6d = nd - 7d$$

$$\Rightarrow 43 - 23 = 48 - 7d \Rightarrow -7d = 20 - 48 = -28$$

$$\Rightarrow -7d = -28 \Rightarrow d = \frac{-28}{-7} = 4$$

$$\begin{aligned} \text{b) From } A_7, A_7 &= A_1 + (6d) \Rightarrow 23 = A_1 + 6(4) \\ \Rightarrow A_1 &= 23 - 24 = -1 \end{aligned}$$

$$\text{c) Since } nd = 48, d = 4 \Rightarrow n = 12$$

14. Let $\{a_1, a_2, a_3, \dots\}$ be an A.P with $a_m = n$ and $a_n = m$ for $n \neq m$. Find a_k in terms of m and n .

Solution: $a_m = a_1 + (m - 1)d = n$ and $a_n = a_1 + (n - 1)d = m$

$$\Rightarrow a_m - a_n = a_1 + (m - 1)d - (a_1 + (n - 1)d)$$

$$\Rightarrow n - m = d(m - 1 - (n - 1)) = d(m - n) \Rightarrow d = \frac{n - m}{m - n} = -1$$

To find a_1 put $d = -1$ in $a_m = a_1 + (m - 1)d$

$$\Rightarrow a_1 + (m - 1)(-1) = n \Rightarrow a_1 = n + m - 1$$

$$\text{Therefore, } a_k = a_1 + (k - 1)d \Rightarrow a_k = n + m - 1 + (k - 1)(-1)$$

$$\Rightarrow a_k = n + m - k$$

15. In each part, find the common difference of a_n :

$$\text{a) } a_n = 4n + 6 \quad \text{b) } a_n = \frac{3 - 5n}{2} \quad \text{c) } a_n = 7 - 3n$$

$$\begin{aligned} \text{Solution: a) } d &= a_{n+1} - a_n \Rightarrow d = 4(n+1) + 6 - (4n + 6) \\ &= 4n + 4 + 6 - 4n - 6 = 4 \end{aligned}$$

$$\text{b) } d = a_{n+1} - a_n \Rightarrow d = \frac{3 - 5(n+1) - (3 - 5n)}{2}$$

$$= \frac{3 - 5n - 5 - 3 + 5n}{2} = \frac{-5}{2}$$

$$\text{c) } d = a_{n+1} - a_n \Rightarrow d = 7 - 3(n+1) - (7 - 3n) = 7 - 3n - 3 - 7 + 3n = -3$$

$$b) \quad A_n = A_1 + (n-1)d \Rightarrow 6 + (n-1)8 = 258$$

$$\Rightarrow 8n - 2 = 258 \Rightarrow 8n = 260 \Rightarrow n = \frac{260}{8} = \frac{65}{2} \notin \mathbb{N}$$

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$$\Rightarrow A_p - A_q = d(p-1) - d(q-1) = d(p-1-q+1) = d(p-q)$$

$$\Rightarrow q - p = d(p-q) \Rightarrow d = \frac{p-q}{q-p} = -1$$

$$\Rightarrow A_q = A_1 + (q-1)(-1) = p \Rightarrow A_1 = p + q - 1$$

$$\Rightarrow A_{(p+q)} = A_1 + (p+q-1)d = p+q-1 + (p+q-1)(-1)$$

$$\Rightarrow A_{(p+q)} = p+q-1-p-q+1 = 0$$

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Solution: $A_n = A_1 + (n-1)d$

$$A_{2n} = A_1 + (2n-1)d$$

$$A_n - A_{2n} = (n-1)d - (2n-1)d$$

$$\Rightarrow x - y = d(n-1-2n+1) = -nd \Rightarrow nd = y - x$$

$$\text{But } A_n = A_1 + (n-1)d = A_1 + nd - d$$

$$\Rightarrow x = A_1 + (y-x) - d \Rightarrow A_1 - d = 2x - y$$

$$\text{And } A_{3n} = A_1 + (3n-1)d = A_1 + 3nd - d = A_1 - d + 3nd$$

$$\Rightarrow A_{3n} = 2x - y + 3(y-x) = -x + 2y$$

13. The 7th term of an A.P is 23, the n^{th} term is 43 and $(2n)^{\text{th}}$ term is

91. Find: a) common difference b) A_1 c) n

Solution: $A_{2n} = A_1 + (2n - 1)d$ and $A_n = A_1 + (n - 1)d = 43$

a) $A_{2n} - A_n = d(2n - 1 - (n - 1)) = nd$
 $\Rightarrow 91 - 43 = nd \Rightarrow nd = 48$

Also, $A_7 - A_1 = nd - d - 6d = nd - 7d$
 $\Rightarrow A_n - A_7 = nd - d - 6d = nd - 7d$

$\Rightarrow 43 - 23 = 48 - 7d \Rightarrow -7d = 20 - 48 = -28$

$\Rightarrow -7d = -28 \Rightarrow d = \frac{-28}{-7} = 4$

b) From A_7 , $A_7 = A_1 + (6d) \Rightarrow 23 = A_1 + 6(4)$
 $\Rightarrow A_1 = 23 - 24 = -1$

c) Since $nd = 48$, $d = 4 \Rightarrow n = 12$

14. Let $\{a_1, a_2, a_3, \dots\}$ be an A.P with $a_m = n$ and $a_n = m$ for $n \neq m$. Find a_k in terms of m and n .

Solution: $a_m = a_1 + (m - 1)d = n$ and $a_n = a_1 + (n - 1)d = m$

$\Rightarrow a_m - a_n = a_1 + (m - 1)d - (a_1 + (n - 1)d)$

$\Rightarrow n - m = d(m - 1 - (n - 1)) = d(m - n) \Rightarrow d = \frac{n - m}{m - n} = -1$

To find a_1 put $d = -1$ in $a_m = a_1 + (m - 1)d$

$\Rightarrow a_1 + (m - 1)(-1) = n \Rightarrow a_1 = n + m - 1$

Therefore, $a_k = a_1 + (k - 1)d \Rightarrow a_k = n + m - 1 + (k - 1)(-1)$

$\Rightarrow a_k = n + m - k$

15. In each part, find the common difference of a_n :

a) $a_n = 4n + 6$ b) $a_n = \frac{3 - 5n}{2}$ c) $a_n = 7 - 3n$

Solution: a) $d = a_{n+1} - a_n \Rightarrow d = 4(n+1) + 6 - (4n + 6)$
 $= 4n + 4 + 6 - 4n - 6 = 4$

b) $d = a_{n+1} - a_n \Rightarrow d = \frac{3 - 5(n+1) - (3 - 5n)}{2}$
 $= \frac{3 - 5n - 5 - 3 + 5n}{2} = \frac{-5}{2}$

c) $d = a_{n+1} - a_n \Rightarrow d = 7 - 3(n+1) - (7 - 3n) = 7 - 3n - 3 - 7 + 3n = -3$

16. In each part, find i) common difference
ii) n^{th} term of an A.P

a) $A_{20} = -1$ and $A_{30} = -6$ c) $A_1 = 8$ and $A_6 = 23$
b) $A_7 = 32$ and $A_{12} = 57$ d) $A_5 = -13$ and $a_8 = -25$

Solution: a) $A_{20} = A_1 + (20 - 1)d = -1 \Rightarrow A_1 + 19d = -1$
 $\Rightarrow A_{30} = A_1 + 29d = -6 \Rightarrow A_1 + 29d = -6$
 $\Rightarrow A_{20} - A_{30} = (A_1 + 19d) - (A_1 + 29d)$
 $\Rightarrow -1 - (-6) = -10d \Rightarrow d = \frac{-1}{2}$

$\therefore$ common difference $d = \frac{-1}{2}$

To find A_1 , put $d = \frac{-1}{2}$ in $A_{20} = A_1 + 19d = -1$

$\Rightarrow A_1 + 19\left(\frac{-1}{2}\right) = -1 \Rightarrow A_1 = -1 + \frac{19}{2} = \frac{17}{2}$

The n^{th} term, $A_n = \frac{17}{2} + (n-1)\left(\frac{-1}{2}\right) = 9 - \frac{1}{2}n$

b) $A_7 = A_1 + 6d = 32 \Rightarrow A_1 + 6d = 32$ and $A_{12} = A_1 + 11d = 57$
 $\Rightarrow A_{12} - A_7 = A_1 + 11d - (A_1 + 6d)$
 $\Rightarrow 57 - 32 = 11d - 6d \Rightarrow 5d = 25 \Rightarrow d = 5$

$\therefore$ common difference, $d = 5$

To find A_1 , put $d = 5$ in $A_7 = A_1 + 6d = 32$

$\Rightarrow A_1 + 6(5) = 32 \Rightarrow A_1 = 32 - 30 = 2$

$\therefore$ The n^{th} term $A_n = 2 + (n-1)5 = 5n - 3$

c) and (d) exercise left for you (Tip)

17. A man accepts a position with an initial salary of Birr 14,400 per year. If it is known that his salary will increase at the end of every year by birr 1,440. what will be his annual salary:

- a) at the beginning of the 11th year
b) at the beginning of 5th year
c) at the end of 5th year

Solution: $A_1 = 14,400$, $d = 1,440$

The beginning of the 11th year is the same as the end of the 10th year.

a) $A_{10} = A_1 + 9d = 14,400 + 9(1,440) = 27,360$

⇒ His annual salary at the beginning of the 11th year is 27,360 birr

Beginning of 5th year is the same as end of 4th year

b) $\Rightarrow A_4 = A_1 + 3d = 14,400 + 3(1,440) = 18,720$

⇒ His annual salary at the beginning of 5th year is 18,720

$A_5 = A_1 + 4d \Rightarrow 14,400 + 4(1,440) = 20,160$

c) $\Rightarrow$ His annual salary at the end of 5th year is birr 20,160

8. A person is scheduled to get a rise of Birr 50 every 6 months, during his/her 5 years on the job. If his/her starting salary is Birr 1600 per month what will his/her annual salary be at the end of 3rd year.

Solution: Let A_1 = his annual salary

$\Rightarrow A_1 = 1600 \times 12 = 19,200$ birr per year

$\Rightarrow d = 50$ birr

The number (period) of increment, in 3 year is $n = 2 \times 3 = 6$

$\Rightarrow$ At the end of 3rd year (A_7)

$\Rightarrow A_7 = A_1 + 6d = 19,200 + 6(50) = \text{Birr } 19500$

19. Find the 25th term of an arithmetic sequence

a) 5, 9, 13, 17

b) 6, 10, 14, 18

Solution: a) $A_n = A_1 + (n - 1)d \Rightarrow A_{25} = A_1 + 24d$

$\Rightarrow A_{25} = 5 + 24(4) = 5 + 96 = 101$

b) $A_{25} = 6 + 24(4) = 6 + 96 = 102 \leftarrow 25^{\text{th}} \text{ term}$

Arithmetic Mean between a and b.

The terms of an arithmetic sequence that are between two given terms are called **arithmetic means** of the given terms.

• If $\{a, C_1, C_2, C_3, \dots, C_k, b\}$ is an arithmetic sequence then

$C_1, C_2, C_3, \dots, C_k$ are k-arithmetic mean between a and b.

• $b = (k + 2)^{\text{th}}$ term of A.P

Illustrative Example

20. a) Insert 3 arithmetic mean between 11 and 19

b) Insert 4 arithmetic mean between 2 and 3

c) Insert n arithmetic mean between 2 and 3

d) Insert n arithmetic mean between a and b

Solution: a) Let, m_1, m_2, m_3 be the 3 arithmetic means

11, $m_1, m_3, 19$ are A.P

$d = \frac{b - a}{k + 1} = \frac{19 - 11}{4} = \frac{8}{4} = 2$

$$\begin{aligned} \Rightarrow A_5 &= 19 \Rightarrow A_1 + 4d = 19 \\ \Rightarrow 11 + 4d &= 19 \Rightarrow 4d = 19 - 11 = 8 \\ \Rightarrow 4d &= 8 \Rightarrow d = 2 \\ \Rightarrow m_1 &= A_2 = A_1 + d = 11 + 2 = 13 \\ \Rightarrow m_1 &= 13, m_2 = 15, m_3 = 17 \end{aligned}$$

$\therefore \{13, 15, 17\}$ are 3 arithmetic mean between 11 and 19

b) 2, $m_1, m_2, m_3, m_4, 3$ are A.P

$$\Rightarrow A_6 = A_1 + 5d = 3 \Rightarrow 2 + 5d = 3 \Rightarrow d = \frac{3-2}{5} = \frac{1}{5}$$

$$\Rightarrow m_1 = A_2 = A_1 + d = 2 + \frac{1}{5} = \frac{11}{5}$$

$$\Rightarrow m_2 = A_3 = A_1 + 2d = 2 + 2\left(\frac{1}{5}\right) = \frac{12}{5}$$

$$\Rightarrow m_3 = A_4 = A_1 + 3d = 2 + 3\left(\frac{1}{5}\right) = \frac{13}{5}$$

$$\Rightarrow m_4 = A_5 = A_1 + 4d = 2 + 4\left(\frac{1}{5}\right) = \frac{14}{5}$$

$\therefore \left\{\frac{11}{5}, \frac{12}{5}, \frac{13}{5}, \frac{14}{5}\right\}$ are 4 arithmetic mean between 2 and 3

c) 2, $m_1, m_2, m_3, \dots, m_n, 3$

$$\Rightarrow A_{n+2} = 3 \Rightarrow A_{n+2} = A_1 + (n+2-1)d \Rightarrow 2 + (n+1)d = 3$$

$$\Rightarrow (n+1)d = 3 - 2 \Rightarrow d = \frac{1}{n+1} \leftarrow \text{common difference}$$

$$\Rightarrow m_1 = 2 + d = 2 + \frac{1}{n+1} = \frac{2n+3}{n+1}$$

$$\Rightarrow m_2 = 2 + 2d = 2 + \frac{2}{n+1} = \frac{2n+4}{n+1}$$

$$\Rightarrow m_n = 2 + nd = 2 + n\left(\frac{1}{n+1}\right) = 2 + \frac{n}{n+1}$$

$$= \frac{2n+2+n}{n+1} = \frac{3n+2}{n+1}$$

Explanation

$$d = \frac{A_5 - A_1}{5-1} = \frac{19-11}{4} = 2$$

Explanation

$$d = \frac{A_6 - A_1}{6-1} = \frac{3-2}{5} = \frac{1}{5}$$

d) $a, m_1, m_2, m_3, \dots, m_n, b$ are n arithmetic mean of A.P
 between a and $b \Rightarrow A_1 = a, A_2 = m_1, \dots, A_{n+2} = b$
 $\Rightarrow A_{n+2} = A_1 + (n+2-1)d = A_1 + (n+1)d = b$ but $A_1 = a$
 $\Rightarrow a + (n+1)d = b \Rightarrow (n+1)d = b - a$
 $\Rightarrow d = \frac{b-a}{n+1}$ $\leftarrow$ common difference terms of a and b .

$$\Rightarrow m_1 = a + d = a + \frac{b-a}{n+1}$$

$$\{m_1, m_2, m_3, \dots, m_n\} = \left\{ a + \frac{(b-a)}{n+1}, a + 2 \frac{(b-a)}{n+1}, \dots, a + \frac{n(b-a)}{n+1} \right\}$$

Sigma notation and partial sum of A.P

A series is the indicated sum of the terms of a sequence. (series is the sum of terms of given sequence.)

If $a_1, a_2, a_3, a_4, \dots, a_n$ are terms of sequence)

Thus $S_1 = a_1 \leftarrow$ sum of first term.

$S_2 = a_1 + a_2 \leftarrow$ sum of the first two terms of sequence

$S_3 = a_1 + a_2 + a_3 \leftarrow$ sum of the first three terms of sequence

$S_4 = a_1 + a_2 + a_3 + a_4 \leftarrow$ sum of the first four terms of a_n sequence

$\vdots$

$S_n = a_1 + a_2 + a_3 + \dots + a_n \leftarrow$ sum of the first n terms of sequence

$S = a_1 + a_2 + a_3 + \dots + a_n \dots \leftarrow$ sum of all the terms of sequence

The sigma (Σ) notation

The Greek letter (Σ) is used to write all the term of infinite and finite series

$$i) S_2 = \sum_{k=1}^2 a_k = a_1 + a_2$$

$$ii) S_3 = \sum_{k=1}^3 a_k = a_1 + a_2 + a_3$$

$$iii) S_4 = \sum_{k=1}^4 a_k = a_1 + a_2 + a_3 + a_4$$

$$iv) S_n = \sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \dots + a_n \leftarrow \text{sum of } n^{\text{th}} \text{ partial sum}$$

Λ) $S = \sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots + a_n + \dots \leftarrow$ sum of all not partial sum. (Infinite series)

• In this notation, the letter k is the **index of summation**, i is the lower limit,

•
$$\sum_{k=1}^n C a_k = C \sum_{k=1}^n a_k$$

•
$$\sum_{k=1}^n (a_k \pm b_k) = \sum_{k=1}^n a_k \pm \sum_{k=1}^n b_k$$

•
$$\sum_{k=1}^n C = C + C + C + \dots + C = nC$$

•
$$\sum_{k=m}^n C = (n - m + 1)C$$

Example: a)
$$\sum_{k=1}^7 7 = 7 + 7 + 7 + 7 + 7 = 5 \times 7 = 35$$

b)
$$\sum_{k=4}^9 7 = (9 - 4 + 1)7 = 42$$

c)
$$\sum_{k=15}^{28} 8 = (28 - 15 + 1)8 = 14 \times 8 = 112$$

Illustrative Example

21. Find the sum (k^{th} partial sum)

a)
$$\sum_{k=1}^5 k$$

d)
$$\sum_{k=4}^{11} 12$$

b)
$$\sum_{k=3}^9 (5k - 4)$$

e)
$$\sum_{n=1}^8 \log_3 \frac{n+1}{n}$$

c)
$$\sum_{k=2}^6 3k$$

f)
$$\sum_{n=1}^{12} \left(\frac{3}{n} - \frac{3}{n+1} \right)$$

Solution: a)
$$\sum_{k=1}^5 k = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{k=3}^9 (5k-4) = 5 \sum_{k=3}^9 k - \sum_{k=3}^9 4$$

$$= 5(3+4+5+6+7+8+9) - (9-3+1)4 = 210 - 20 = 190$$

$$\sum_{k=2}^6 3k = 3(2+3+4+5+6) = 60$$

$$\sum_{k=4}^{11} 12 = (11-4+1)12 = 96$$

$$\sum_{n=2}^9 \log_3 n \Rightarrow S_8 = \log_3 2 + \log_3 3 + \log_3 4 + \log_3 5 + \log_3 6 + \dots + \log_3 9$$

$$= \log_3 \left(\frac{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}{3} \right) = \log_3 9 = 2$$

$$\bullet S_{10} = \log_3^{11} \quad \bullet S_n = \log_3^{(n+1)}$$

$$\bullet S_{26} = \log_3^{27} = 3$$

$$\sum_{r=1}^n \left(\frac{3}{n} - \frac{3}{n+1} \right) = 3 \sum_{n=1}^n \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 3 \left[\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \right]$$

$$S_n = 3 \left(1 - \frac{1}{n+1} \right) = 3 - \frac{3}{n+1}$$

$$\bullet S_{12} = 3 - \frac{3}{12+1} = \frac{39-3}{13} = \frac{36}{13}$$

$$\bullet S_{100} = 3 - \frac{3}{100} = \frac{297}{100} = 2.97$$

$$\bullet S_{200} = 3 - \frac{3}{201} = 3 \left(1 - \frac{1}{201} \right) = 3 \left(\frac{200}{201} \right) = \frac{600}{201}$$

Note: $(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots$ is called **telescoping**
series a_2 is canceled by 2nd term, a_3 is canceled by 3rd term and so
on so that, the n^{th} partial sum of this series is $S_n = a_1 - a_{n+1}$

22. Find the indicated partial sum (S_n)

a) $a_n = \frac{1}{n} - \frac{1}{n+1}$, S_{10} , S_{50} , S_{100} , S_n

b) $a_n = \ln\left(\frac{n+1}{n}\right)$, S_n , S_{100} , S_{200} , ...

c) $a_n = \frac{1}{n+2} - \frac{1}{n+3}$, S_n , S_{20} , S_{50} , S_{100} , S_{200}

Solution: a) $\sum_{n=1}^n a_n = \sum_{n=1}^n \left(\frac{1}{n} - \frac{1}{n+1} \right)$
 $= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right)$
 $\Rightarrow S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1} \Rightarrow S_{10} = 1 - \frac{1}{11} = \frac{11-1}{11} = \frac{10}{11}$

$\Rightarrow S_{50} = \frac{50}{51}$, $S_{100} = \frac{100}{101}$

b) $\sum_{n=1}^n a_n = \sum_{n=1}^n \ln\left(\frac{n+1}{n}\right) = \ln 2 + \ln \frac{3}{2} + \ln \frac{4}{3} + \dots + \ln \frac{n+1}{n}$
 $= \ln \left(2 \times \frac{3}{2} \times \frac{4}{3} \times \dots \times \frac{n}{n-1} \times \frac{n+1}{n} \right) = \ln(n+1)$

$\Rightarrow S_n = \ln(n+1) \Rightarrow S_{100} = \ln(101) \Rightarrow S_{200} = \ln(201)$

c) $\sum_{n=1}^n a_n = \sum_{n=1}^n \left(\frac{1}{n+2} - \frac{1}{n+3} \right) = \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \dots + \left(\frac{1}{n+2} - \frac{1}{n+3} \right)$

$\Rightarrow S_n = \frac{1}{3} - \frac{1}{n+3} = \frac{n+3-3}{3(n+3)} = \frac{n}{3(n+3)}$

$\Rightarrow S_{20} = \frac{20}{3(23)} = \frac{20}{69} \Rightarrow S_{50} = \frac{50}{3(50+3)} = \frac{50}{159}$

$\Rightarrow S_{100} = \frac{100}{3(100+3)} = \frac{100}{309}$

Write the series using sigma notation

a) $8 + 11 + 14 + \dots + 98$

b) $6 + 11 + 16 + \dots + 161$

c) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{30}{31}$

d) $7 + 12 + 17 + \dots + 5n + 2$

Solution: First determine the n^{th} term

a) $a_1 = 8, d = 3 \Rightarrow a_n = 8 + (n-1)3 = 3n + 5$

$\Rightarrow 98 = 3n + 5 \Rightarrow 3n = 93 \Rightarrow n = 31$

$$S_{31} = \sum_{n=1}^{31} 3n + 5 = 8 + 11 + 14 + \dots + 98$$

b) $a_1 = 6, d = 5 \Rightarrow a_n = 6 + (n-1)5 = 5n + 1$

$\Rightarrow a_n = 161 \Rightarrow 5n + 1 = 161 \Rightarrow n = \frac{160}{5} = 32$

$$\Rightarrow 6 + 11 + 16 + \dots + 161 = S_{32} = \sum_{n=1}^{32} 5n + 1$$

c) $S_{30} = \sum_{n=1}^{30} \frac{n}{n+1}$

d) $S_n = 7 + 12 + 17 + \dots + 5n + 2 = \sum_{k=1}^n 5k + 2$

Express $\sum_{k=3}^7 5^{k-2}$ in sigma notation with $k = -1$ as the lower limit of summation.

Solution: $\sum_{k=3}^7 5^{k-2} = 5^1 + 5^2 + 5^3 + 5^4 + 5^5$

$$= 5^{-1+2} + 5^{0+2} + 5^{1+2} + 5^{2+2} + 5^{3+2} = \sum_{k=-1}^3 5^{k+2}$$

$$\Rightarrow \sum_{k=2}^7 5^{k-2} = \sum_{k=-1}^3 5^{k+2}$$

$$= \sum_{k=4}^n k^2 = 4^2 + 5^2 + 6^2 + \dots + n^2$$

b) $k = -5$ as lower limit

Solution: $\sum_{k=4}^n k^2 = 4^2 + 5^2 + 6^2 + \dots + n^2$

$$= (0+4)^2 + (1+4)^2 + (2+4)^2 + (3+4)^2 = \sum_{k=0}^3 (k+4)^2$$

b) $\sum_{k=-5}^n k^2 = 4^2 + 5^2 + 6^2 + \dots + n^2$

$$= (-1+5)^2 + (0+5)^2 + (1+5)^2 + (2+5)^2$$

$$= \sum_{k=-1}^2 (k+5)^2$$

Sum of Arithmetic Progressions

- Consider the following arithmetic sum written in **increasing order** and then in **decreasing order**. Paired vertically and added. The sums form a new series in which all the addends are the same

$$\begin{array}{r} S_1 = 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 \\ S_2 = 27 + 24 + 21 + 18 + 15 + 12 + 9 + 6 \end{array}$$

$$\hline 2S_1 = 33 + 33 + 33 + 33 + 33 + 33 + 33 + 33$$

$$\Rightarrow 2S_1 = 8(33) = 264 \Rightarrow S_1 = 132$$

This method generates n^{th} partial sum of A.P.

Let $\{A_k\}_{k=1}^n$ be an A.P. and S_n be the sum of the first n terms of an A.P.

That is $S_n = A_1 + A_2 + A_3 + \dots + A_n$

$$\begin{array}{r} S_n = A_1 + (A_1 + d) + (A_1 + 2d) + \dots + A_n - d + A_n \end{array}$$

$$\begin{array}{r} S_n = A_n + (A_n - d) + (A_n - 2d) + \dots + A_1 + d + A_1 \end{array}$$

$$\hline 2S_n = (A_1 + A_n) + (A_1 + A_n) + (A_1 + A_n) + \dots + (A_1 + A_n)$$

$$\Rightarrow 2S_n = n(A_1 + A_n), \text{ but } A_n = A_1 + (n-1)d$$

$$\Rightarrow S_n = \frac{n}{2}(A_1 + A_n) = \frac{n}{2}(2A_1 + (n-1)d)$$

25. Express $\sum_{k=4}^7 k^3$ in sigma notation with a) $k = 0$ as lower limit
b) $k = -1$ as lower limit

Solution: a) $\sum_{k=4}^7 k^3 = 4^3 + 5^3 + 6^3 + 7^3$

$$= (0+4)^3 + (1+4)^3 + (2+4)^3 + (3+4)^3 = \sum_{k=0}^3 (k+4)^3$$

b) $\sum_{k=4}^7 k^3 = 4^3 + 5^3 + 6^3 + 7^3$

$$= (-1+5)^3 + (0+5)^3 + (1+5)^3 + (2+5)^3$$

$$= \sum_{k=-1}^2 (k+5)^3$$

Sum of Arithmetic Progressions

- Consider the following arithmetic sum written in **increasing order** and then in decreasing order. Paired vertically and added. The sums form a new series in which all the addends are the same

$$+ \begin{cases} S_8 = 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 \\ S_8 = 27 + 24 + 21 + 18 + 15 + 12 + 9 + 6 \end{cases}$$

$$2S_8 = 33 + 33 + 33 + 33 + 33 + 33 + 33 + 33$$

$$\Rightarrow 2S_8 = 8(33) = 264 \Rightarrow S_8 = 132$$

This method generates n^{th} partial sum of A.P.

Let $\{A_n\}_{n=1}^{\infty}$ be an A.P., and S_n be the sum of the first n terms of an A.P.

That is $S_n = A_1 + A_2 + A_3 + \dots + A_n$

$$+ \begin{cases} S_n = A_1 + (A_1 + d) + (A_1 + 2d) + \dots + A_n - d + A_n \\ S_n = A_n + (A_n - d) + (A_n - 2d) + \dots + A_1 + d + A_1 \end{cases}$$

$$2S_n = (A_1 + A_n) + (A_1 + A_n) + (A_1 + A_n) + \dots + (A_1 + A_n)$$

$$\Rightarrow 2S_n = n(A_1 + A_n), \text{ but } A_n = A_1 + (n-1)d$$

$$\Rightarrow S_n = \frac{n}{2}(A_1 + A_n) = \frac{n}{2}(2A_1 + (n-1)d)$$

The n^{th} partial sum of an arithmetic sequence

$\{A_n\}_{n=1}^{\infty}$ with common difference d is

$$S_n = \frac{n(A_1 + A_n)}{2}, \text{ ii) } S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

Illustrative Example

Ex. Find the sum of the first:

- 100 natural number
- 100 odd natural number
- 50 even natural number
- 100 natural number exactly divisible by 7
- 100 terms of A.P between 3 and 4
- n -natural number
- n -odd natural number
- n -even natural number

Solution: we use $S_n = \frac{n}{2}(A_1 + A_n)$

a) $1 + 2 + 3 + \dots + 100 \Rightarrow S_{100} = \frac{100}{2}(1 + 100) = 5,050$

b) The n^{th} odd natural number is $A_n = 2n - 1$
 $\Rightarrow$ the 100^{th} odd natural number is
 $\Rightarrow A_{100} = 2(100) - 1 = 199 \Rightarrow A_1 = 1, A_{100} = 199$
 $\Rightarrow S_{100} = \frac{100}{2}(1 + 199) = (50)(200) = 10,000$

c) We can use $S_n = \frac{n}{2}[2A_1 + (n-1)d] \Rightarrow A_1 = 2, d = 2$
 $\Rightarrow S_{50} = \frac{50}{2}(2(2) + (50-1)(2)) = 25(4 + 98) = 2,500$

d) The sequence is $\{7, 14, 21, \dots\}$, $S_n = \frac{n}{2}[2A_1 + (n-1)d]$
 $\Rightarrow A_1 = 7, d = 7 \Rightarrow S_{100} = \frac{100}{2}(2(7) + (99)(7)) = 35,350$

$$e) \quad A_1 = 3, A_{100} = 4, \Rightarrow S_{100} = \frac{100}{2} [3 + 4] = 50 \times 7 = 350$$

$$f) \quad n^{\text{th}} \text{ partial sum is denoted by } S_n = \frac{n}{2} [2A_1 + (n-1)d]$$

$$\Rightarrow S_n = \frac{n}{2} (2(1) + (n-1)1) = \frac{n^2 + n}{2}$$

$$g) \quad A_1 = 1, d = 2 \Rightarrow S_n = \frac{n}{2} (2(1) + (n-1)2) = \frac{n}{2} (2n) = n^2$$

$$h) \quad A_1 = 2, d = 2 \Rightarrow$$

$$S_n = \frac{n}{2} (2(2) + (n-1)2) = \frac{n}{2} (2n + 2) = n^2 + n$$

27. Find the sum of two digit numbers which leave the remainder when divided by 3.

Solution: by division algorithm, $A_n = 3n + 2$

$$\Rightarrow 11, 14, 17, \dots, 98$$

$$\Rightarrow 98 = 11 + (n-1)3 \Rightarrow 3n = 90 \Rightarrow n = 30$$

$$\text{Use, } S_n = \frac{n}{2} (A_1 + A_n) \Rightarrow S_{30} = \frac{30}{2} (11 + 98) = 15(109) = 1635$$

28. Find the sum of the sequence

$$a) \quad 18 + 21 + 24 + \dots + 90 \quad b) \quad 12 + 19 + 26 + \dots + 89$$

Solution: First determine the number of terms

$$a) \quad A_n = A_1 + (n-1)d \Rightarrow 90 = 18 + (n-1)3 \Rightarrow 3n = 75$$

$$\Rightarrow n = 25$$

$$\Rightarrow S_{25} = \frac{25}{2} (18 + 89) \Rightarrow 1350$$

$$b) \quad A_n = A_1 + (n-1)d \Rightarrow 89 = 12 + (n-1)7 \Rightarrow 7n = 84 \Rightarrow n = 12$$

$$\Rightarrow S_{12} = \frac{12}{2} (12 + 89) \Rightarrow 606$$

29. Find the sum of the first 20 terms of $2 + 5 + 8 + \dots$

$$\text{Solution: } S_{20} = \frac{20}{2} (2(2) + (20-1)3) = 10 \times 61 = 610$$

Find the sum of all 3 digit numbers between 200 and 400 which are divisible by 13.

Solution: The least 3 digit numbers greater than 200 divisible by 13 is 208 $\Rightarrow A_1 = 208$, $A_n = 390$

$$\text{To find } n, A_n = A_1 + (n-1)d \Rightarrow 390 = 208 + (n-1)13$$

$$\Rightarrow 13n = 390 - 208 + 13 = 195$$

$$\Rightarrow 13n = 195 \Rightarrow n = \frac{195}{13} = 15$$

$$\Rightarrow S_n = \frac{15}{2}(208 + 390) = 4485$$

If $A_5 = 15$ and $A_{24} = 91$, then find

a) S_{20} b) S_{30} c) S_n

Solution: a) $S_n = \frac{n}{2}(2A_1 + (n-1)d) \Rightarrow S_{20} = \frac{20}{2}(2A_1 + 19d)$

To find d , $A_{24} - A_5 = (A_1 + 23d) - (A_1 + 4d)$

$$\Rightarrow 91 - 15 = 19d \Rightarrow d = \frac{76}{19} = 4$$

To find A_1 , $A_5 = A_1 + 4d \Rightarrow 15 = A_1 + 4(4) \Rightarrow A_1 = 15 - 16 = -1$

$$\Rightarrow S_{20} = \frac{20}{2}(2(-1) + (20-1)4) = \frac{20}{2}(74) = 740$$

b) $S_{30} = \frac{30}{2}(2(-1) + (30-1)4) = 15(-2 + 116) = 1710$

c) $S_n = \frac{n}{2}(2(-1) + (n-1)4) = 2n^2 - 3n$

Find the maximum partial sum

Consider the following

Find the maximum sum

a) $9 + 6 + 3 + 0 + -3 + -6 \dots$

b) $13 + 10 + 7 + 4 + 1 + -2 + -5 + \dots$

Here a) $9 + 6 + 3 + 0 = 18 \leftarrow$ maximum sum

b) $13 + 10 + 7 + 4 + 1 = 35 \leftarrow$ maximum sum

Note: Maximum sum is obtained when the n^{th} term ($A_n \geq 0$) (the sum is maximum if we add only all positive term)

Illustrative Example

32. Find the maximum sum:

a) $A_1 = 42, d = -5$

c) $A_3 = 96, d = -8$

b) $A_1 = 19, d = -2$

d) $A_{10} = 70, A_{18} = 46$

Solution: a) A.P. = 42, 37, 32, ...

Set; n^{th} term $\geq 0 \Rightarrow A_n \geq 0 \Rightarrow (A_1 + (n-1)d) \geq 0$

$$\Rightarrow (42 + (n-1)(-5)) \geq 0 \Rightarrow 47 \leq 5n \Rightarrow n \leq 9\frac{2}{5}$$

Since n is natural number, $\Rightarrow n = 9$

$$\Rightarrow S_9 = \frac{9}{2}(2(42) + (9-1)(-5)) = 198 \leftarrow \text{maximum sum.}$$

b) A.P. = 19, 17, 15, ... $\Rightarrow n^{\text{th}}$ term $\geq 0 \Rightarrow A_n \geq 0$

$$\Rightarrow (19 + (n-1)(-2)) \geq 0 \Rightarrow -2n + 21 \leq 0 \Rightarrow n \leq 10\frac{1}{2}$$

$$\Rightarrow n = 10$$

$$\Rightarrow S_{10} = \frac{10}{2}(2(19) + (9)(-2)) = 100 \leftarrow \text{maximum sum}$$

c) $A_3 = 96, d = -8 \Rightarrow A_3 = A_1 + 2d \Rightarrow 96 = A_1 + 2(-8)$

$$\Rightarrow A_1 = 112, \text{ set: } A_n \geq 0 \Rightarrow (112 + (n-1)(-8)) \geq 0$$

$$\Rightarrow 120 - 8n \geq 0 \Rightarrow n \geq 15 \Rightarrow n = 15$$

$$\therefore S_{15} = \frac{15}{2}(2(112) + (15-1)(-8)) = 840 \leftarrow \text{maximum sum of sequence}$$

d) $A_{10} = 70$ and $A_{18} = 46$

To find d , $A_{18} - A_{10} = (A_1 + 17d) - (A_1 + 9d)$

$$\Rightarrow 46 - 70 = 8d \Rightarrow d = \frac{-24}{8} = -3$$

To find A_1 , $A_{10} = A_1 + 9d \Rightarrow 70 = A_1 + 9(-3)$

$$\Rightarrow A_1 = 70 + 27 = 97$$

Set, $A_n \geq 0 \Rightarrow (A_1 + (n-1)(-3)) \geq 0 \Rightarrow (97 + -3n + 3) \geq 0$

$$\Rightarrow 100 - 3n \geq 0 \Rightarrow n \geq 33\frac{1}{3} \Rightarrow n = 33$$

$$\therefore S_{33} = \frac{33}{2}(97 + 1) = 1617 \leftarrow \text{maximum sum of sequence}$$

33. Find S_6 for the given arithmetic sequence

a) $A_1 = 6, d = 3$

b) $A_1 = 5, d = 4$

c) $A_{10} = 11, d = 2$

Solution: use $S_n = \frac{n}{2}(2A_1 + (n-1)d)$

a) $S_6 = \frac{6}{2}(2(6) + (5)(3)) = 3(27) = 81$

b) $S_6 = \frac{6}{2}(2(5) + 5(4)) = 90$

c) **First, find A_1 , $A_{10} = A_1 + 9d \Rightarrow 11 = A_1 + 18 \Rightarrow A_1 = -7$**

$$\Rightarrow S_6 = \frac{6}{2}(2(-7) + 5(2)) = -12$$

34. Find the indicated term of an arithmetic sequence

a) $S_{10} = 210, A_1 = 3$, Find A_{10}

b) $S_{20} = 600, A_{16} = 41$, find A_1, A_{20}, A_n .

c) $S_{10} = 145, A_9 = 25$, find S_9, d, A_n .

d) $S_n = 300, A_1 = 3, d = 4$, find n, A_n

Solution: a) $S_{10} = \frac{10}{2}(A_1 + A_{10})$

$$\Rightarrow 210 = 5(3 + A_{10}) \Rightarrow A_{10} + 3 = 42 \Rightarrow A_{10} = 39$$

b) $A_{16} = A_1 + 15d \Rightarrow 41 = A_1 + 15d \Rightarrow A_1 = 41 - 15d$

$$S_{20} = \frac{20}{2}(2A_1 + 19d) \Rightarrow 600 = 10[2(41 - 15d) + 19d]$$

$$\Rightarrow 60 = 82 - 30d + 19d$$

$$\Rightarrow 60 - 82 = -11d \Rightarrow d = \frac{-22}{-11} = 2$$

So that $A_1 = 41 - 15d = 41 - 15(2) = 11$

$$\Rightarrow A_{20} = A_1 + 19d = 11 + 19(2) = 49$$

$$\Rightarrow A_n = A_1 + (n-1)d = 11 + (n-1)(2) = 2n + 9$$

c) $A_9 = A_1 + 8d \Rightarrow 25 = A_1 + 8d \Rightarrow A_1 = 25 - 8d$

$$\Rightarrow S_{10} = \frac{10}{2}[2(25 - 8d) + 9d] \Rightarrow 145 = 5[50 - 16d + 9d]$$

$$\Rightarrow 29 = 50 - 7d \Rightarrow -7d = -21 \Rightarrow d = \frac{21}{7} = 3$$

$$\therefore A_1 = 25 - 8(3) = 25 - 24 = 1$$

$$\bullet S_9 = \frac{9}{2}[A_1 + A_9] = \frac{9}{2}[1 + 25] = 117$$

$$\bullet A_n = A_1 + (n-1)d \Rightarrow A_n = 1 + (n-1)3 = 3n-2$$

$$\Rightarrow n(2n+1) = 300 \Rightarrow 2n^2 + n - 300 = 0$$

$$\therefore n = \frac{-1 \pm \sqrt{1 - 4(2)(-300)}}{2(2)} = \frac{-1 \pm \sqrt{2401}}{4} = \frac{-1 \pm 49}{4} = 12$$

$$\Rightarrow A_n = A_1 + (n-1)d \Rightarrow A_n = 3 + (n-1)4 = 4n-1$$

35. If $S_8 = 12$ and $S_{16} = 56$, then find

a) Common difference (d)

b) the n^{th} term

Solution:

$$S_8 = \frac{8}{2}[2A_1 + 7d] \Rightarrow \begin{cases} 2A_1 + 7d = 3 \\ 2A_1 + 15d = 7 \end{cases} \Rightarrow d = \frac{4}{8} = \frac{1}{2}$$

$$S_{16} = \frac{16}{2}[2A_1 + 15d] \Rightarrow \frac{16}{2}[2A_1 + 15d] = 56 \Rightarrow 2A_1 + 15d = 7$$

$$\text{b) From } S_8, 2A_1 + 7\left(\frac{1}{2}\right) = 3 \Rightarrow A_1 = \frac{-1}{4}$$

$$\Rightarrow A_n = A_1 + (n-1)d \Rightarrow A_n = \frac{-1}{4} + (n-1)\left(\frac{1}{2}\right) = \frac{2n-3}{4}$$

36. Find the sum of 12 terms of the A.P. -7, -3, -1, ... starting from the 8th term

$$\text{Solution: } a_1 = A_8 = A_1 + 7d = -7 + 7(4) = 21$$

$$a_{12} = A_{19} = A_1 + 18d = -7 + 18(4) = 65$$

$$\Rightarrow S_{12} = \frac{12}{2}[a_1 + a_{12}] = 6[21 + 65] = 6(86) = 516$$

37. The sum of the first 10 terms of an A.P is 15 and the sum of next 10 terms is 215.

Find a) d

b) the general term (n^{th} term)

$$\text{Solution: } S_{10} = \frac{n}{2}(2A_1 + 9d) \Rightarrow 15 = \frac{10}{2}(2A_1 + 9d)$$

$$\text{a) } 2A_1 + 9d = 3 \dots \dots \dots *$$

$$S_{20} = \frac{20}{2}(2A_1 + 19d)$$

$$\Rightarrow 15 + 215 = 10(2A_1 + 19d) \Rightarrow 2A_1 + 19d = 23 \dots **$$

Solving (*) and (**)

$$\begin{cases} 2A_1 + 9d = 3 \\ 2A_1 + 19d = 23 \end{cases}$$

$$-10d = -20 \Rightarrow d = 2$$

$$b) \Rightarrow 2A_1 + 9d = 3 \Rightarrow 2A_1 + 18 = 3 \Rightarrow A_1 = \frac{-15}{2}$$

$$\Rightarrow A_n = A_1 + (n-1)d = \frac{-15}{2} + (n-1)(2) = 2n - \frac{19}{2}$$

$$\Rightarrow \text{general term (n}^{\text{th}} \text{ term), } A_n = \frac{4n-19}{2}$$

38. Find S_n (n^{th} partial sum) of arithmetic sequence.

$$a) \quad a_n = 3n + 2 \quad b) \quad a_n = 4n - 3 \quad c) \quad a_n = 1 - 2n$$

Solution: a) $a_1 = 3(1) + 2 = 5$, $a_n = 3n + 2$

$$\Rightarrow S_n = \frac{n}{2}[a_1 + a_n] = \frac{n}{2}[5 + 3n + 2] = \frac{3n^2 + 7n}{2}$$

$$b) \quad S_n = \frac{n}{2}[a_1 + a_n] = \frac{n}{2}[1 + 4n - 3] = 2n^2 - n$$

$$c) \quad S_n = \frac{n}{2}[a_1 + a_n] = \frac{n}{2}[-1 + 1 - 2n] = -n^2$$

39. Find the n^{th} term (a_n), common difference (d) for the given n^{th} partial sum of arithmetic sequence.

$$a) \quad S_n = 2n^2 + 3n \quad b) \quad S_n = 2n^2 + n \quad c) \quad S_n = 3n^2 - 5n$$

$$\text{Solution: } S_n = \underbrace{a_1 + a_2 + a_3 + \dots + a_{n-1}}_{S_{n-1}} + a_n \Rightarrow a_n = S_n - S_{n-1}$$

$$a) \quad a_n = S_n - S_{n-1} \Rightarrow a_n = (2n^2 + 3n) - (2(n-1)^2 + 3(n-1))$$

$$\Rightarrow a_n = (2n^2 + 3n) - (2(n-1)^2 + 3(n-1))$$

$$\Rightarrow a_n = 2n^2 + 3n - 2n^2 + 4n + 1 - 3n + 3 = 4n + 1$$

$$\Rightarrow a_1 = 5, a_2 = 9 \Rightarrow d = a_2 - a_1 = 9 - 5 = 4$$

$$b) \quad a_n = S_n - S_{n-1} \Rightarrow a_n = (2n^2 + n) - (2(n-1)^2 + (n-1))$$

$$\Rightarrow a_n = 2n^2 + n - 2n^2 + 4n - 2 + -n + 1 = 4n - 1 \Rightarrow d = 4$$

c) exercise left for you (Tip!)

40. If $S_n = n^2 + 3n - 2$, find
 a) 5th term b) 12th term c) $(n+1)$ th term

Solution: $a_n = s_n - s_{n-1}$

$$\text{a) } a_5 = s_5 - s_4 = (5^2 + 3(5) - 2) - (4^2 + 3(4) - 2) \\ \Rightarrow a_5 = 25 + 15 - 2 - 16 - 12 + 2 = 12$$

$$\text{b) } a_{12} = s_{12} - s_{11} \Rightarrow a_{12} = (144 + 36 - 2) - (121 + 33 - 2) = 26$$

$$\text{c) } a_{n+1} = s_{n+1} - s_n \Rightarrow a_{n+1} = ((n+1)^2 + 3(n+1) - 2) - (n^2 + 3n - 2) \\ \Rightarrow a_{n+1} = n^2 + 5n + 4 - n^2 - 3n + 2 = 2n + 6$$

41. Find

$$\text{a) } \sum_{n=1}^{30} (3n-1) \quad \text{b) } \sum_{n=1}^{50} (2n+5) \quad \text{c) } \sum_{n=1}^{50} (n+2) \\ \text{d) } \sum_{n=1}^{80} \log_3 \frac{n}{n+1} \quad \text{e) } \sum_{k=n+1}^{2n} k \quad \text{f) } \sum_{n=0}^{59} x^3$$

Solution: $S_n = \frac{n}{2} [A_1 + A_n]$

$$\text{a) } S_{30} = \frac{30}{2} [A_1 + A_{30}] = 15[1 + 89] = 1350$$

$$\text{b) } S_{50} = \frac{50}{2} [A_1 + A_{50}] = 25[7 + 105] = 2800$$

$$\text{c) } S_{60} = \frac{60}{2} [A_1 + A_{60}] = 30[3 + 62] = 1950$$

$$\text{d) } \sum_{n=1}^{80} \log_3 \frac{n}{n+1} = \log_3 \frac{1}{2} + \log_3 \frac{2}{3} + \log_3 \frac{3}{4} + \dots + \log_3 \frac{n-1}{n} + \log_3 \frac{n}{n+1}$$

$$= \log_3 \left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{n-1}{n} \times \frac{n}{n+1} \right)$$

$$\Rightarrow S_n = \log_3 \frac{1}{n+1} \Rightarrow S_n = -\log_3^{n+1} \Rightarrow S_{80} = -\log_3^{81} = -4$$

$$\text{e) } A_1 = n + 1, A_n = 2n$$

$$S_n = \frac{n}{2} [A_1 + A_n] = \frac{n}{2} [n + 1 + 2n] = \frac{n}{2} [3n + 1]$$

$$\text{f) } \text{Since } x^3 \text{ is constant with respect to lower limit } n$$

$$\Rightarrow \sum_{n=0}^{59} x^3 = (59 - 0 + 1)x^3 = 60x^3$$

42. Find k : a) $\sum_{n=1}^k (2n+5) = 520$ b) $\sum_{n=1}^{10} (4n-k) = 190$

Solution: a) $S_k = \frac{k}{2}[A_1 + A_k] \Rightarrow \frac{k}{2}(7 + 2k + 5)$

$$\Rightarrow k^2 + 6k = 520 \Rightarrow k^2 + 6k - 520 = 0$$

$$\Rightarrow (k-20)(k+26) = 0 \Rightarrow k = 20$$

$$\therefore k = 20, \text{ i.e. } \sum_{n=1}^{20} (2n+5) = 520$$

b) $S_{10} = \frac{10}{2}[A_1 + A_{10}] \Rightarrow 190 = 5[4 - k + 40 - k]$

$$\Rightarrow 44 - 2k = 38 \Rightarrow -2k = 38 - 44 = -6 \Rightarrow -2k = -6 \Rightarrow k = 3$$

$$\therefore k = 3, \text{ i.e. } \sum_{n=1}^{10} (4n-3) = 190$$

Sum of arithmetic mean between a and b

Let $m_1, m_2, m_3, \dots, m_n$ be n -arithmetic means between a and b

$$\Rightarrow a, m_1, m_2, m_3, \dots, m_n, b$$

So that, $m_1 + m_2 + m_3 + \dots + m_n = \frac{n}{2}[m_1 + m_n]$

$$\bullet m_1 = a + d \text{ and } b - m_n = d \Rightarrow m_n = b - d$$

$$\therefore m_1 + m_2 + m_3 + \dots + m_n = \frac{n}{2}[a + d + b - d] = \frac{n}{2}[a + b]$$

Illustrative Example

43. Find the sum of:

- 3 arithmetic mean between 3 and 4
- 10 arithmetic mean between 7 and 13
- n arithmetic mean between 3 and 4
- n arithmetic mean between 4 and 6
- n arithmetic mean between a and b

Solution:

$$\Rightarrow \sum_{p=0}^{59} x^3 = (59 - 0 + 1)x^3 = 60x^3$$

42. Find k: a) $\sum_{n=1}^k (2n+5) = 520$ b) $\sum_{n=1}^{10} (4n-k) = 190$

Solution: a) $S_k = \frac{k}{2}[A_1 + A_k] \Rightarrow \frac{k}{2}(7 + 2k + 5)$

$$\Rightarrow k^2 + 6k = 520 \Rightarrow k^2 + 6k - 520 = 0$$

$$\Rightarrow (k-20)(k+26) = 0 \Rightarrow k = 20$$

$$\therefore k = 20, \text{ i.e. } \sum_{n=1}^{20} (2n+5) = 520$$

b) $S_{10} = \frac{10}{2}[A_1 + A_{10}] \Rightarrow 190 = 5[4 - k + 40 - k]$

$$\Rightarrow 44 - 2k = 38 \Rightarrow -2k = 38 - 44 = -6 \Rightarrow -2k = -6 \Rightarrow k = 3$$

$$\therefore k = 3, \text{ i.e. } \sum_{n=1}^{10} (4n-3) = 190$$

Sum of arithmetic mean between a and b

Let $m_1, m_2, m_3, \dots, m_n$ be n -arithmetic means between a and b

$$\Rightarrow a, m_1, m_2, m_3, \dots, m_n, b$$

So that, $m_1 + m_2 + m_3 + \dots + m_n = \frac{n}{2}[m_1 + m_n]$

• $m_1 = a + d$ and $b - m_n = d \Rightarrow m_n = b - d$

$$\therefore m_1 + m_2 + m_3 + \dots + m_n = \frac{n}{2}[a + d + b - d] = \frac{n}{2}[a + b]$$

Illustrative Example

43. Find the sum of:

- 3 arithmetic mean between 3 and 4
- 10 arithmetic mean between 7 and 13
- n arithmetic mean between 3 and 4
- n arithmetic mean between 4 and 6
- n arithmetic mean between a and b

Solution:

$$\text{b) } S_{10} = \frac{10}{2}[7 + 13] = \frac{(10)(20)}{2} = 100$$

c) $S_n = \frac{n}{2}[3+4] = \frac{7n}{2}$

$$d) \quad S_n = \frac{n}{2}[4+6] = 5n$$

e) $S_n = \frac{n}{2}[a+b]$

- a) the value of n b) common difference

Solution: $11 + m_1 + m_2 + m_3 + \dots + m_n + 19 = 150$

$$S_k = \frac{k}{2}[A_1 + A_k] = \frac{k}{2}[11 + 19] = 150 \Rightarrow k = 10$$

$$\Rightarrow n = (\text{total number of term}) - (\text{two end point term})$$

$$\Rightarrow n = k - 2 \Rightarrow n = 10 - 2 = 8$$

$$b) \quad A_{10} = A_1 + 9d \Rightarrow 19 = 11 + 9d \Rightarrow 9d = 8 \Rightarrow d = \frac{8}{9}$$

$$\therefore \text{common difference, } d = \frac{8}{9}$$

45. How many terms of the series may be taken their sum
 $65 - 7 + -4 + -1 + 2 \dots$

Solution: Let $S_n = 65$, $n = ?$

$$\Rightarrow S_n = \frac{n}{2}[2(-7) + (n-1)(3)] \Rightarrow 65 = \frac{n}{2}[3n-3-14]$$

$$\Rightarrow 3n^2 - 17n = 130 \Rightarrow 3n^2 - 17n - 130 = 0$$

$$\Rightarrow 3n^2 - 30n + 13n - 130 \Rightarrow 3n(n-10) + 13(n-10) = 0$$

$$\Rightarrow (3n + 13)(n - 10) = 0 \Rightarrow n = 10$$

46. A man saved birr 16,500 in ten years. In each year after the first he saved birr 100 more than he did in the preceding year. How much did he save in the first year.

Solution: $S_{10} = 16,500$, $d = 100$

48?

$$\Rightarrow S_{10} = \frac{10}{2}[2A_1 + 9d] \Rightarrow 16500 = 5[2A_1 + 900]$$

$$\Rightarrow 2A_1 - 900 = 3,300 \Rightarrow 2A_1 = 2,400$$

$$\therefore A_1 = 1200 \text{ birr} \leftarrow \text{saving in the first year}$$

47. A man saves birr 500 of his salary in certain year and in each subsequent year he will save birr 150 more than the previous year. How much will he save during the ninth year

Solution:

$$A_1 = 500, \quad d = 150 \Rightarrow A_9 = A_1 + 8d = 500 + 8(150) = 1700$$

48. **Theater seating:** The seating section in a theater hall has total of 10 rows. The first row has 27 seats, the second row has 29 seats, and the third row has 31 seats, and so on.

- How many seats are there in the tenth row.
- How many seats are there in the section (seating capacity of hall)

Solution: $A_1 = 27, d = 2$

a) $A_{10} = A_1 + (9d) = 27 + 9(2) = 45 \text{ seats}$

b) $S_{10} = \frac{n}{2}[A_1 + A_{10}] = \frac{10}{2}(27 + 45) = 360 \text{ total seats}$

49. **Contest prize:** A contest offers 15 prizes. The first prize is worth birr 5,000 and each successive prize is worth birr 250 less than the preceding prize. Find the value of the fifteenth prize and the total value of the prize

Solution: $A_1 = 5,000 \quad d = -250$

$$\Rightarrow A_{15} = A_1 + 14d \Rightarrow 5,000 + 14(-250) = 5000 - 3500 = 1500$$

$$\Rightarrow S_{15} = \frac{15}{2}[5000 + 1500] = 48,750$$

50. **Prize money:** A contest will have 5 prizes totaling birr 5000 and there will be a birr 100 difference between successive prizes. Find the first prize

Solution: $S_5 = 5000, d = -100$

$$A_5 = A_1 + 4d = A_1 + 4(-100) = A_1 - 400$$

$$\Rightarrow S_5 = \frac{5}{2}[A_1 + A_5] = \frac{5}{2}[A_1 + A_1 - 400]$$

$$\Rightarrow 5000 = \frac{5}{2}[2A_1 - 400] \Rightarrow 2A_1 - 400 = 2000$$

$$\Rightarrow 2A_1 = 2400 \Rightarrow A_1 = \frac{2400}{2} = 1200$$

51. Find $\sum_{n=1}^{60} (-1)^n (2n+3)$

Solution: $-5 + 7 - 9 + 11 \dots + -121 + 123$.

Explanation: $7 + 11 + 15 + \dots + 123 \leftarrow$ for even terms

$$\Rightarrow a_n = a_1 + (n-1)d \Rightarrow a_n = 7 + (n-1)4 = 4n+3$$

$-5 - 9 - 13 - 17 \dots - 121 \leftarrow$ for odd terms

$$\Rightarrow a_n = -5 + (n-1)(-4) = -4n-1$$

$$\Rightarrow \sum_{n=1}^{60} (-1)^n (2n+3) = \sum_{n=1}^{30} (-4n-1) + \sum_{n=1}^{30} (4n+3)$$

$$\Rightarrow S_{60} = S_{30} + S_{30}^1 = \frac{30}{2} [-5 + -121] + \frac{30}{2} [7 + 123]$$

$$= -1890 + 1950 = 60$$

$$\therefore \sum_{n=1}^{60} (-1)^n (2n+3) = 60$$

52. Elon Academy offers a job applicants with choice of two salary increase plans.

With firm A, starting salary of birr 10,000 with increase of birr 1000 per year.

With firm B, offers starting salary of birr 12,000 with increase of birr 600 per year.

- What would the annual salary be in the 9th year at firm A at firm B?
- Over the first 9 year, how much would be earned at firm A, at firm B?
- Which firm should you choose?

Solution: a) The annual salary in both firm (plan) forms arithmetic sequence

With firm A(plan A): 10,000, 11,000, 12,000....

$$\Rightarrow A_1 = 10,000, \quad d = 1000$$

$$\Rightarrow A_9 = A_1 + 8d \Rightarrow A_9 = 10,000 + 8(1000) = 18,000$$

With firm B, 12,000, 12,600, 13,200...

$$\Rightarrow A_1 = 12,000, \quad d = 600$$

$$\Rightarrow A_9 = a_1 + 8d = 12,000 + 8(600) = 16,800$$

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b) over the first 9 year

$$\text{with firm A: } S_9 = \frac{9}{2}(10,000 + 18,000) = 126,000$$

$$\text{with firm B: } S_9 = \frac{9}{2}(12,000 + 16,800) = 129,600$$

c) Firm B offers more than firm A by 3600.

Geometric sequence

• A geometric sequence is characterized by a common ratio (r) between successive term.

• Let $\{G_n\}$ be a geometric sequence if and only if.

$$\Rightarrow \frac{G_2}{G_1} = \frac{G_3}{G_2} = \frac{G_4}{G_3} = \dots = \frac{G_{n+1}}{G_n} = r, \quad r \in \mathbb{R} \setminus \{0\}$$

$$\Rightarrow G_2 = G_1 r, G_3 = G_2 r = G_1 r^2, \boxed{G_n = G_1 r^{n-1}}$$

Illustrative Example

§3. If $G_1 = 9$, $r = \frac{1}{3}$, then find

a) G_6

b) n^{th} term

Solution:

$$\text{a) } G_6 = G_1 r^5 = (9) \left(\frac{1}{3} \right)^5 = \frac{1}{27}$$

$$\text{b) } G_n = G_1 r^{n-1} = (9) \left(\frac{1}{3} \right)^{n-1} = 9(3^{1-n})$$

§4. If $G_3 = 5$, and $G_6 = -40$

Find a) common ratio (r) b) G_1 c) G_8

Solution: $G_6 = G_1 r^5$ and $G_3 = G_1 r^2$

$$\Rightarrow \frac{G_6}{G_3} = \frac{G_1 r^5}{G_1 r^2} = \frac{-40}{5} \Rightarrow r^3 = -8 \Rightarrow r = -2 \leftarrow \text{common ratio}$$

$$\text{b) } G_8 = G_1 r^7 = \left(\frac{5}{4} \right) (-2)^7 = -160$$

§5. Find the indicated term of geometric sequence

$$\text{a) } G_3 = \frac{16}{3} \text{ and } G_5 = \frac{64}{27}, \text{ Find } G_1 \text{ and } r$$

- b) $G_4 = 3$ and $G_7 = 24$, find G_1 , r and G_n
 c) $G_2 = 2 + \sqrt{2}$ and $G_5 = 4\sqrt{2} + 4$, Find r , G_1 and G_n
 d) $G_4 = 3$ and $G_6 = 9$, Find r and G_1
 e) $G_4 = 4$ and $G_7 = 12$, Find r and G_1 and G_{10}

Solution: a) $G_3 = G_1 r^2$ and $G_5 = G_1 r^4$

$$\Rightarrow \frac{G_5}{G_3} = \frac{G_1 r^4}{G_1 r^2} = \frac{64}{27} \cdot \frac{3}{16} \Rightarrow r^2 = \frac{4}{9} \Rightarrow r = \pm \frac{2}{3}$$

$$\Rightarrow G_3 = G_1 r^2 \Rightarrow \frac{16}{3} = G_1 \left(\frac{2}{3} \right)^2 \Rightarrow G_1 = \frac{16}{3} \cdot \frac{9}{4} = 12$$

- b) $G_4 = G_1 r^3$ and $G_7 = G_1 r^6$

$$\Rightarrow \frac{G_7}{G_4} = \frac{G_1 r^6}{G_1 r^3} = \frac{24}{3} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

$$\Rightarrow G_4 = G_1 r^3 \Rightarrow 3 = G_1 (2)^3 \Rightarrow G_1 = \frac{3}{8}$$

- c) $G_2 = G_1 r$ and $G_5 = G_1 r^4$

$$\Rightarrow \frac{G_5}{G_2} = \frac{G_1 r^4}{G_1 r} \Rightarrow r^3 = \frac{4\sqrt{2} + 4}{2 + \sqrt{2}} = 2\sqrt{2} = \sqrt{8} \Rightarrow r = \sqrt{2}$$

$$\Rightarrow G_2 = G_1 r \Rightarrow 2 + \sqrt{2} = G_1 (\sqrt{2}) \Rightarrow G_1 = \frac{2 + \sqrt{2}}{\sqrt{2}} = \sqrt{2} + 1$$

$$\Rightarrow G_{11} = G_1 r^{10} = (\sqrt{2} + 1)(\sqrt{2})^{10} = 32\sqrt{2} + 32$$

- d) $G_4 = G_1 r^3$ and $G_6 = G_1 r^5$

$$\Rightarrow \frac{G_6}{G_4} = \frac{G_1 r^5}{G_1 r^3} \Rightarrow r^2 = \frac{9}{3} = 3 \Rightarrow r = \pm \sqrt{3}$$

$$\Rightarrow G_4 = G_1 r^3 \Rightarrow 3 = G_1 (\pm \sqrt{3})^3 \Rightarrow G_1 = \frac{3}{\pm \sqrt{27}} = \frac{3}{\pm 3\sqrt{3}} = \frac{1}{\pm \sqrt{3}}$$

- e) $G_4 = G_1 r^3$ and $G_7 = G_1 r^6$

$$\Rightarrow \frac{G_7}{G_4} = \frac{G_1 r^6}{G_1 r^3} \Rightarrow r^3 = \frac{12}{4} = 3 \Rightarrow r = \sqrt[3]{3}$$

$$\Rightarrow G_4 = G_1 r^3 \Rightarrow 4 = G_1(3) \Rightarrow G_1 = \frac{4}{3}$$

$$\Rightarrow G_{10} = G_1 r^9 \Rightarrow G_{10} = \left(\frac{4}{3}\right)(\sqrt[3]{3})^9 = 36$$

If $x+1$, $x+3$ and $x+7$ are consecutive terms of a G.P. Find the value of x .

Solution: $\frac{x+3}{x+1} = \frac{x+7}{x+3} \Rightarrow (x+3)^2 = (x+1)(x+7)$

$$\Rightarrow x^2 + 6x + 9 = x^2 + 8x + 7 \Rightarrow 8x - 6x = 9 - 7$$

$$\Rightarrow 2x = 2 \Rightarrow x = 1, \text{ G.P.: } 2, 4, 8.$$

Find the term of the sequence:

a) $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \dots$ which is less than $\frac{1}{1,024}$

b) $5, 50, 500, \dots$ which exceeds 500,000.

Solution: a) $G_n = G_1 r^{n-1} \Rightarrow G_n = (4)\left(\frac{1}{2}\right)^{n-1} < 2^{-10}$

$$\Rightarrow 2^{3-n} < 2^{-10} \Leftrightarrow 3-n < -10 \Rightarrow -n < -13 \Rightarrow n > 13$$

$\therefore$ The smallest value of $n = 14$

b) $G_n = G_1 r^{n-1} \geq 500,000 \Rightarrow (5)(10)^{n-1} \geq 500,000$

$$\Rightarrow 10^{n-1} \geq 100,000 \Rightarrow 10^{n-1} \geq 10^5 \Leftrightarrow n-1 \geq 5$$

$$\Rightarrow n \geq 6$$

$\therefore$ The smallest value of $n = 6$.

Growth of bacteria: A certain culture initially contains 10,000 bacteria and increase by 20% every hour. Find the number of bacteria after

a) t hours

b) 10 hours

Solution: The population growth at the end of n -hour is given by

$$G_n = [1 + \text{rate}]^n G_1$$

a) $G_t = G_1 [1 + 0.2]^t = 10,000 [1.2]^t$

b) $G_{10} = (10,000) [1.2]^{10} \approx 61,917$

Population growth: The population of a certain town is increasing at the rate of 5% per year. If the present number of population is 48,000. Find the population after

a) t year

b) 4 year

c) 10 year

Solution: $G_t = G_1 (1 + \text{rate})^t$

a) $G_t = 48,000 (1 + 0.05)^t = 48,000 (1.05)^t$

$$b) \quad G_4 = (48,000)(1.05)^4 \approx 58,344.$$

$$c) \quad G_{10} = (48,000)(1.05)^{10} \approx 78,186.$$

61. Hana deposits birr 6000 in a bank account that earns an annual interest rate of 8%. Find the amount she has at end of

- a) the first year c) the 4th year
b) the second year d) the nth year

Solution: The amount at the end of n year is given by

$$G_n = G_1[1 + \text{rate}]^n$$

- a) At the first year = $6,000(1 + 0.08) = 6480$
b) $G_2 = 6000(1 + 0.08)^2 = 6,998$
c) $G_4 = 6000(1 + 0.08)^4 = 6,000(1.08)^4 \approx 8,162$
d) $G_n = 6,000(1.08)^n$

62. If the value of Lifan car depreciated by 25% per year. If its present value is birr 240,000. What will be the value at the end of
- a) 2 year b) 8 year c) nth year

Solution: The depreciation value at the end of t year is given by

$$G_t = G_1(1 - \text{rate})^t$$

- a) $G_2 = 240,000(1 - 0.25)^2 = 240,000(0.75)^2 = 135,000$
b) $G_8 = 240,000(1 - 0.25)^8 = 240,000(0.75)^8 = 24,027$
c) $G_n = 240,000(0.75)^n$

63. A car that is bought for birr 120,000 depreciate in value by birr 8000 per year. How long will it take for the car to make lose of 20% of its value?

Solution: Let the cost of the car at the end of n year (A_n and G_n)

- Arithmetic depreciation is $A_n = 120,000 - 8000n$
- Geometric depreciation is $G_n = 120,000(1 - 0.2)$

$$\Rightarrow A_n = G_n \Rightarrow 120,000 - 8000n = 96,000$$

$$\Rightarrow 120,000 - 96,000 = 8,000n \Rightarrow 24,000 = 8,000n \Rightarrow n = 3$$

$\therefore$ It takes 3 year to make loss of 20% of its value.

Geometric Mean

If a and b are positive real number then a positive number c is called the **geometric mean** of a and b if a, c, b is geometric sequence.

$$\Rightarrow r = \frac{c}{a} = \frac{b}{c} \Rightarrow c^2 = ab \Rightarrow c = \sqrt{ab}$$

Product of n geometric mean between a and b is $(\sqrt{ab})^n = (ab)^{\frac{n}{2}}$

Find the geometric mean for each pair:

- a) 3 and 12 b) -3 and -27 c) $\sqrt{5} + \sqrt{2}$ and $\sqrt{5} - \sqrt{2}$

Solution: a) $G = \sqrt{(3)(12)} = 6$

b) $G = -\sqrt{(-3)(-27)} = -\sqrt{81} = -9$

c) $G = \sqrt{(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})} = \sqrt{5 - 2} = \sqrt{3}$

For two positive numbers **a** and **b** show that the arithmetic mean between **a** and **b** is greater than or equal to their geometric mean.

Solution: Arithmetic mean, $A = \frac{1}{2}(a + b)$

Geometric mean, $G = \sqrt{ab}$

To show $A \geq G \Rightarrow A - G \geq 0$

$$\Rightarrow \frac{1}{2}(a + b) - \sqrt{ab} \geq 0 \Rightarrow a + b - 2\sqrt{ab} \geq 0$$

$$\Rightarrow (\sqrt{a} - \sqrt{b})^2 \geq 0 \Rightarrow a + b - 2\sqrt{ab} \geq 0$$

$$\Rightarrow \frac{a + b}{2} \geq \sqrt{ab} \Rightarrow A \geq G$$

66. Insert four arithmetic mean and 5 geometric mean between 3 and 9.

Solution: Let m_1, m_2, m_3, m_4 are four A.M

$$\Rightarrow 3, m_1, m_2, m_3, m_4, 9 \text{ are 6 A.P}$$

$$\text{Common diff. } d = \frac{A_6 - A_1}{5} = \frac{9 - 3}{5} = \frac{6}{5}$$

$$\Rightarrow \text{A.M} = 3 + \frac{6}{5}, 3 + \frac{12}{5}, 3 + \frac{18}{5}, 3 + \frac{24}{5}$$

When 5 geometric means are inserted between 3 and 9, then

$$G_1 = G_1 r^6$$

$$\Rightarrow 9 = 3r^6 \Rightarrow r^6 = \frac{9}{3} = 3 \Rightarrow r = \sqrt[6]{3}$$

$$\Rightarrow \text{G.M} = 3\sqrt[6]{3}, 3\sqrt[3]{3}, 3\sqrt{3}, 3\sqrt[3]{9}, 3\sqrt[6]{3^5}$$

Note: If $\{a_n\}$ is geometric sequence then $\{\log a_n\}$ is arithmetic sequence.

Find common difference

Solution: a_n is geometric

$$\Rightarrow \frac{a_2}{a_1} = \frac{a_3}{a_2} = \dots = \frac{a_n}{a_{n-1}} = r \leftarrow \text{common ratio}$$

For A.P, $\{\log a_n\} = \log a_1, \log a_2, \log a_3 \dots \log a_n$

$$\Rightarrow \log a_2 - \log a_1 = \log a_3 - \log a_2 = \dots \log a_n - \log a_{n-1}$$

$$\Rightarrow \log \left(\frac{a_2}{a_1} \right) = \log \left(\frac{a_3}{a_2} \right) = \log(r) \leftarrow \text{common difference}$$

Example: 2, 4, 8, 16, ... G.P with $r = 2$

$\log 2, \log 4, \log 8, \log 16, \dots$ A.P

$$\Rightarrow \log 4 - \log 2 = \log 8 - \log 4 = \dots = d$$

$$\Rightarrow \log \frac{4}{2} = \log \frac{8}{4} = \log 2$$

If $A_1, A_2, A_3, \dots$ and A_n are n arithmetic mean between any two real numbers a and b and $G_1, G_2, G_3, \dots$ and G_n are ' n ' geometric mean between a and b then:

$$\text{i) } \frac{A_1 + A_2}{G_1 G_2} = \frac{a + b}{ab} \longrightarrow$$

$$\text{ii) } \frac{A_1 + A_2 + A_3}{G_1 \cdot G_2 \cdot G_3} = \frac{\frac{3}{2}(a+b)}{\sqrt{(ab)^3}}$$

$$\text{iii) } \frac{A_1 + A_2 + A_3 + \dots + A_n}{G_1 \cdot G_2 \cdot G_3 \dots G_n} = \frac{\frac{n}{2}(a+b)}{\sqrt{(ab)^n}}$$

Explanation

$$A_1 - a = b - A_2$$

$$\Rightarrow A_1 + A_2 = a + b$$

$$\frac{G_1}{a} = \frac{b}{G_2}$$

$$\Rightarrow G_1 G_2 = ab$$

Sum of Geometric progression

The n^{th} partial sum S_n of a geometric sequence with first term G_1 and common ratio r is given by

$$S_n = \begin{cases} nG_1, & \text{if } r = 1 \\ \frac{G_1(1-r^n)}{1-r} = \frac{G_1(r^n-1)}{r-1}, & \text{if } r \neq 1 \end{cases}$$

$$\text{To proof: } S_n = G_1 + G_1r + G_1r^2 + \dots + G_1r^{n-2} + G_1r^{n-1}$$

$$rS_n = rG_1 + G_1r^2 + G_1r^3 + \dots + G_1r^{n-1} + G_1r^n$$

Subtracting rS_n from S_n

$$\Rightarrow S_n - rS_n = G_1 - G_1r^n \Rightarrow S_n(1-r) = G_1(1-r^n)$$

$$\Rightarrow S_n = \frac{G_1(1-r^n)}{1-r} = \frac{G_1(r^n-1)}{r-1}$$

Illustrative Example

Find the indicated sum:

a) $G_1 = -2$, $r = 3$, Find S_6

b) $G_3 = 9$, and $G_5 = \frac{81}{4}$, Find r , G_1 and S_4

c) $G_4 = 24$, and $G_7 = 192$, Find r , G_1 and S_{10}

Solution: Here

$$a) S_6 = \frac{G_1(r^6-1)}{r-1} = \frac{-2(3^6-1)}{3-1} = -728$$

$$b) G_3 = G_1 r^2 \text{ and } G_5 = G_1 r^4 \Rightarrow \frac{G_5}{G_3} = \frac{G_1 r^4}{G_1 r^2} = \frac{81}{4(9)}$$

$$\Rightarrow r^2 = \frac{9}{4} \Rightarrow r = \pm \frac{3}{2}, \text{ Let } r = \frac{3}{2}$$

$$G_3 = G_1 r^2 \Rightarrow 9 = G_1 \left(\frac{9}{4} \right) \Rightarrow G_1 = 4$$

$$\Rightarrow S_4 = \frac{G_1(r^4-1)}{r-1} = 4 \left[\frac{\left(\frac{3}{2} \right)^4 - 1}{\frac{3}{2} - 1} \right] = \frac{65}{2}$$

$$c) \frac{G_7}{G_4} = \frac{G_1 r^6}{G_1 r^3} = \frac{192}{24} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

$$G_4 = G_1 r^3 \Rightarrow 24 = G_1 (8) \Rightarrow G_1 = 3$$

$$S_{10} = \frac{G_1[r^{10}-1]}{r-1} = \frac{3(2^{10}-1)}{2-1} = 3,069$$

Find the partial sum of each geometric sequence

$$a) \sum_{n=1}^4 5(3^{n-1}) \quad b) \sum_{n=1}^7 64\left(\frac{-1}{2}\right)^{n-1} \quad c) \sum_{n=1}^4 (2)^n 3$$

Solution: a) $G_1 = 5, r = 3 \Rightarrow S_4 = \sum_{n=1}^4 5(3^{n-1}) = \frac{5[3^4 - 1]}{3 - 1} = 200$

b) $G_1 = 64, r = \frac{-1}{2} \Rightarrow S_7 = \frac{G_1[r^7 - 1]}{r - 1}$

$$= \frac{64\left(\frac{-1}{128} - 1\right)}{\frac{-1}{2} - 1} = 64\left(\frac{129}{128}\right) \frac{2}{3} = 43$$

c) $G_1 = 6, r = 2 \Rightarrow S_4 = \frac{6[2^4 - 1]}{2 - 1} = 6(15) = 90$

Infinite Series

- **An infinite series** is the sum of an infinite sequence of numbers

$$\{a_n\}_{n=1}^{\infty} \text{ is denoted by } \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

- If the sum of all terms of the sequence is unique finite number,

then the series $\sum_{n=1}^{\infty} a_n$ is called **convergent**, other wise divergent

- If the sequence is geometric sequence, we call an **infinite geometric series**.

- If $\{G_n\}_{n=1}^{\infty}$ is a geometric series with common ration r , then the n^{th} partial sum:

$$S_n = \frac{G_1(1-r^n)}{1-r} = \frac{G_1}{1-r} - \frac{G_1 r^n}{1-r}, r \neq 1$$

Convergence of Geometric series

* If $|r| < 1$, the geometric series:

$$G_1 + G_1 r + G_1 r^2 + \dots + G_1 r^n + \dots \text{ converges to the sum } S = \frac{G_1}{1-r}$$

$1 - r^n \rightarrow 0$

$$\sum_{n=1}^{\infty} G_1 r^{n-1} = \frac{G_1}{1-r}, |r| < 1$$

If $|r| \geq 1$, the series diverges.

Illustrative example

Find the sum of the infinite geometric series

a) $2 + 1 + \frac{1}{2} + \dots$

d) $3 + \sqrt{3} + 1 + \frac{1}{\sqrt{3}} + \dots$

b) $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots$

e) $2 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots$

c) $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \dots$

f) $3 + 0.3 + 0.03 + \dots$

Solution: If $|r| < 1$ then sum $S_{\infty} = \frac{G_1}{1-r}$

a) $G_1 = 2, r = \frac{1}{2} \Rightarrow S_n = \frac{2 \left[1 - \left(\frac{1}{2} \right)^n \right]}{1 - \frac{1}{2}} = 4 \left[1 - \left(\frac{1}{2} \right)^n \right] \leftarrow n^{\text{th}}$

partial sum

As $n \rightarrow \infty, S_{\infty} = \frac{G_1}{1-r} = \frac{2}{1 - \frac{1}{2}} = 4$

b) $G_1 = 2, r = \frac{1}{3}, S_n = \frac{2 \left[1 - \left(\frac{1}{3} \right)^n \right]}{1 - \frac{1}{3}} = 3 \left[1 - \left(\frac{1}{3} \right)^n \right] \leftarrow n^{\text{th}} \text{ partial sum}$

$\therefore S_{\infty} = \frac{G_1}{1-r} = \frac{2}{1 - \frac{1}{3}} = 3 \leftarrow (\text{the sum of all terms of the sequence})$

$$c) G_1 = \frac{1}{3}, r = \frac{-2}{3} \Rightarrow S_\infty = \frac{G_1}{1-r} = \frac{\frac{1}{3}}{1+\frac{2}{3}} = \frac{1}{3} \cdot \frac{3}{5} = \frac{1}{5}$$

$$d) G_1 = 3, r = \frac{1}{\sqrt{3}} \Rightarrow S_\infty = \frac{G_1}{1-r} = \frac{3}{1-\frac{1}{\sqrt{3}}} = \frac{3\sqrt{3}}{\sqrt{3}-1}$$

$$e) 2 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^2} \dots \infty = 2 + \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n, G_1 = \frac{1}{5}, r = \frac{1}{5}$$

$$\Rightarrow S_\infty = 2 + \frac{G_1}{1-r} = 2 + \frac{\frac{1}{5}}{1-\frac{1}{5}} = 2 + \frac{1}{4} = \frac{9}{4}$$

$$f) G_1 = 3, r = \frac{1}{10} \Rightarrow S_\infty = \frac{G_1}{1-r} = \frac{3}{1-\frac{1}{10}} = \frac{3}{1-\frac{1}{10}} = \frac{10}{3}$$

66. In each part, determine whether the series converges, and if so find its sum.

$$a) \sum_{k=0}^{\infty} \left(\frac{5}{4^k}\right)$$

$$e) \sum_{k=1}^{\infty} 3^{k-2}$$

$$b) \sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$$

$$f) \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2}{3}\right)^n$$

$$c) \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k$$

$$g) \sum_{k=1}^{\infty} \frac{3^{k+1}}{4^{k-1}}$$

$$d) \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)\left(\frac{1}{3}\right)^{n-1}$$

$$h) \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} \left(\frac{3}{4}\right)^{k+1}$$

Solution: a) $\sum_{k=0}^{\infty} 5\left(\frac{1}{4^k}\right)$ has $G_1 = 5, r = \frac{1}{4}$

$$r < 1 \Rightarrow \text{converges to } S_{\infty} = \frac{G_1}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$$

$$b) G_1 = 3, r = \frac{1}{2} \Rightarrow r < 1 \Rightarrow \text{convergent.}$$

$$\text{Converges to } S_{\infty} = \frac{G_1}{1-r} = \frac{3}{1-\frac{1}{2}} = 6$$

$$c) G_1 = 1, r = \frac{3}{5} \Rightarrow r < 1, \text{ Converges to } S_{\infty} = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$$

$$d) G_1 = \frac{1}{9}, r = \frac{1}{3} \Rightarrow r < 1, \text{ Converges to } S_{\infty} = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{1}{6}$$

$$e) \sum_{k=1}^{\infty} 3^{k-2} = \sum_{k=1}^{\infty} 3^k \cdot 3^{-2} = \sum_{k=1}^{\infty} 3^k \cdot \frac{1}{9} \Rightarrow G_1 = \frac{1}{9}, r = 3$$

$$\Rightarrow r > 1 \Rightarrow \text{diverge}$$

$\therefore$ the sum doesn't exist.

$$f) \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \left(\frac{2}{3}\right)^n = \frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \frac{16}{81} + \dots$$

$$G_1 = \frac{2}{3}, r = -\frac{2}{3} \Rightarrow \left| -\frac{2}{3} \right| < 1$$

$$\Rightarrow \text{Converges to the sum, } S_{\infty} = \frac{G_1}{1-r} = \frac{\frac{2}{3}}{1+\frac{2}{3}} = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$$

$$g) \sum_{k=1}^{\infty} \frac{3^{k+1}}{4^{k-1}} = \sum_{k=1}^{\infty} \frac{3^1 \cdot 3^k}{4^{-1} \cdot 4^k} = \sum_{k=1}^{\infty} 12 \left(\frac{3}{4}\right)^k$$

$$G_1 = 9, r = \frac{3}{4} \Rightarrow \frac{3}{4} < 1 \quad \therefore \text{Convergent}$$

$$\Rightarrow \text{Converges to the sum, } \frac{G_1}{1-r} = \frac{9}{1-\frac{3}{4}} = 36$$

$$\begin{aligned} \text{h) } \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} \cdot \left(\frac{3}{4}\right)^{k+1} &= \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \cdot \left(\frac{3}{4}\right)^k \cdot \left(\frac{3}{2} \cdot \frac{3}{4}\right) \\ &= \sum_{k=1}^{\infty} \frac{9}{8} \left(\frac{2}{3} \cdot \frac{3}{4}\right)^k \end{aligned}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{9}{8} \left(\frac{1}{2}\right)^k \Rightarrow G_1 = \frac{9}{16}, r = \frac{1}{2} < 1 \quad \therefore \text{Convergent}$$

$$\Rightarrow \text{Converges to the sum, } \frac{G_1}{1-r} = \frac{\frac{9}{16}}{1-\frac{1}{2}} = \frac{9}{8}$$

67. Find the sum

$$\text{a) } \sum_{n=0}^{\infty} \left(\frac{2^n + 5}{3^n}\right)$$

$$\text{c) } \sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$$

$$\text{b) } \sum_{n=1}^{\infty} \left(\frac{3^{n-1} - 1}{6^{n-1}}\right)$$

$$\text{(d) } \sum_{n=1}^{\infty} \left(\frac{4^n - 3^n}{12^n}\right)^2$$

$$\begin{aligned} \text{Solution: a) } \sum_{n=0}^{\infty} \left(\frac{2^n + 5}{3^n}\right) &= \sum_{n=0}^{\infty} \left(\frac{2^n}{3^n}\right) + \sum_{n=0}^{\infty} \left(\frac{5}{3^n}\right) \\ &= \frac{1}{1-\frac{2}{3}} + \frac{5}{1-\frac{1}{3}} = 3 + \frac{15}{2} = \frac{21}{2} \end{aligned}$$

$$\text{b) } \sum_{n=1}^{\infty} \left(\frac{3^{n-1} - 1}{6^{n-1}}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$$

$$= \frac{1}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{6}} = 2 - \frac{6}{5} = \frac{4}{5}$$

$$c) \sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n} \right) = \sum_{n=0}^{\infty} \frac{5}{2^n} - \sum_{n=0}^{\infty} \left(\frac{1}{3^n} \right) = \frac{5}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{3}} = \frac{17}{2}$$

$$\begin{aligned} d) \sum_{n=1}^{\infty} \left(\frac{4^n - 3^n}{12^n} \right)^2 &= \sum_{n=1}^{\infty} \frac{[4^{2n} - 2(4^n \cdot 3^n) + 3^{2n}]}{12^{2n}} \\ &= \sum_{n=1}^{\infty} \left[\left(\frac{4}{12} \right)^{2n} - 2 \left(\frac{12}{144} \right)^n + \left(\frac{3}{12} \right)^{2n} \right] \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{9} \right)^n - 2 \sum_{n=1}^{\infty} \left(\frac{1}{12} \right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{16} \right)^n \\ &= \frac{\frac{1}{9}}{1 - \frac{1}{9}} - 2 \left[\frac{\frac{1}{12}}{1 - \frac{1}{12}} \right] + \frac{\frac{1}{16}}{1 - \frac{1}{16}} = \frac{1}{8} - \frac{2}{11} + \frac{1}{15} = \frac{13}{1320} \end{aligned}$$

68. Find the sum if it exist

a) $\sum_{k=0}^{\infty} \cos(\pi k)$

b) $\sum_{k=0}^{\infty} \frac{\cos(\pi k)}{3^k}$

Solution: $\sum_{k=0}^{\infty} \cos(\pi k) = 1 + (-1) + (1) + (-1) \dots = \begin{cases} 0 \\ 1 \end{cases}$

$$S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Since it is not unique number

$\Rightarrow$ It is divergent, $\therefore$ the sum does not exist

b) $\sum_{k=0}^{\infty} \frac{\cos(\pi k)}{3^k} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots, G_1 = 1, r = -\frac{1}{3}$

$$\therefore S_{\infty} = \frac{G_1}{1-r} = \frac{1}{1 - \left(-\frac{1}{3}\right)} = \frac{1}{1 + \frac{1}{3}} = \frac{3}{4}$$

69. Find the sum of the series: $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \dots$

Solution: $\left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \dots\right)$

$$= \left[\frac{\frac{1}{2}}{1 - \frac{1}{4}} \right] + \left[\frac{\frac{1}{9}}{1 - \frac{1}{9}} \right] = \frac{2}{3} + \frac{1}{8} = \frac{19}{24}$$

A geometric series for a repeating decimal

70. Express the repeating decimal as a fraction using the infinite sum

a) $\frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \dots$

c) $0.2\dot{5}$

b) $0.\dot{8}$

d) $24.\dot{3}$

Solution:

a) $G_1 = \frac{5}{10}, r = \frac{1}{10} \Rightarrow S_{\infty} = \frac{G_1}{1-r} = \frac{\frac{5}{10}}{1 - \frac{1}{10}} = \frac{5}{10} \cdot \frac{10}{9} = \frac{5}{9}$

b) $0.\dot{8} = 0.\dot{8} + 0.0\dot{8} + 0.00\dot{8} + \dots = \frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + \dots$

$$\Rightarrow G_1 = \frac{8}{10}, r = \frac{1}{10} \Rightarrow S_{\infty} = \frac{G_1}{1-r} = \frac{\frac{8}{10}}{1 - \frac{1}{10}} = \frac{8}{10} \cdot \frac{10}{9} = \frac{8}{9}$$

c) $0.2\dot{5} = 0.2 + 0.05 + 0.005 + \dots = 0.2 + \frac{5}{100} + \frac{5}{1000} + \dots$

$$\Rightarrow S_{\infty} = 0.2 + \left[\frac{\frac{5}{100}}{1 - \frac{1}{10}} \right] = 0.2 + \frac{5}{100} \cdot \frac{10}{9} = \frac{2}{10} + \frac{5}{90} = \frac{23}{90}$$

$$d) \quad 24.\dot{3} = 24 + 0.3 + 0.03 + 0.003 + \dots \Rightarrow G_1 = 0.3, r = \frac{1}{10}$$

$$\Rightarrow S_{\infty} = 24 + \left[\frac{\frac{3}{10}}{1 - \frac{1}{10}} \right] = 24 + \frac{3}{10} \cdot \frac{10}{9} = 24 + \frac{1}{3} = 24\frac{1}{3}$$

11. Evaluate in each part of the infinite series

a) $0.2 + 0.22 + 0.222 + 0.2222 + \dots$

b) $0.3 + 0.33 + 0.333 + 0.3333 + \dots$

Solution:

$$a) \quad 0.2 + 0.22 + 0.222 + 0.2222 + \dots = \frac{2}{10} + \frac{22}{100} + \frac{222}{1000} + \dots$$

$$\Rightarrow 2 \left[\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + \dots \right]$$

$$\Rightarrow \frac{2}{9} \left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \dots \right]$$

$$\Rightarrow \frac{2}{9} \left[\frac{10-1}{10} + \frac{10^2-1}{100} + \frac{10^3-1}{1000} + \dots + \frac{10^n-1}{10^n} \right]$$

$$\Rightarrow \frac{2}{9} \left[1 - \frac{1}{10} + 1 - \frac{1}{10^2} + 1 - \frac{1}{10^3} + \dots + 1 - \frac{1}{10^n} \right]$$

$$\Rightarrow \frac{2}{9} \left[(1+1+1+\dots+1) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots + \frac{1}{10^n} \right) \right]$$

$$\Rightarrow \frac{2}{9} \left[n - \frac{\frac{1}{10} \left(1 - \frac{1}{10} \right)^n}{1 - \frac{1}{10}} \right] = \frac{2}{9} \left[n - \frac{1}{9} \left(1 - \left(\frac{1}{10} \right)^n \right) \right]$$

$$\Rightarrow S_n = \frac{2}{9} n - \frac{2}{81} \left(1 - \frac{1}{10^n} \right)$$

b) Exercise left for you.

$$S_n = \frac{3}{9} n - \frac{3}{81} \left(1 - \frac{1}{10^n} \right) = \frac{1}{3} n - \frac{1}{27} \left(1 - \frac{1}{10^n} \right)$$

Geometric series for product of exponential form

72. Find the product

a) $2^1 \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{9}} \cdot 2^{\frac{1}{27}} \dots$ b) $3^x \cdot 3^{x^2} \cdot 3^{x^3} \dots 3^{x^n}$ for $|x| < 1$

Solution: a) $2^1 \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{9}} \cdot 2^{\frac{1}{27}} \dots = 2^{1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots}$

$$= 2^{\frac{G_1}{1-r}} = 2^{\frac{1}{1-\frac{1}{3}}} = 2^{\frac{3}{2}} = \sqrt{2^3} = 2\sqrt{2}$$

b) $3^x \cdot 3^{x^2} \cdot 3^{x^3} \dots = 3^{x+x^2+x^3+\dots+x^n} = 3^{\frac{G_1}{1-r}} = 3^{\frac{x}{1-x}}$

73. Find the value of k;

a) $4^k \cdot 4^{k^2} \cdot 4^{k^3} \dots = 16$ b) $e^1 \cdot e^k \cdot e^{k^2} \cdot e^{k^3} \dots = e^3$

Solution: a) $4^k \cdot 4^{k^2} \cdot 4^{k^3} \dots = 4^{k+k^2+k^3+\dots} = 4^2$

$$\Leftrightarrow k + k^2 + k^3 + \dots = 2 \Rightarrow G_1 = k, r = k$$

Since series convergent

$$\Rightarrow \frac{G_1}{1-r} \Rightarrow \frac{k}{1-k} = 2 \Rightarrow k = 2 - 2k \Rightarrow 3k = 2 \Rightarrow k = \frac{2}{3}$$

b) $e^1 \cdot e^k \cdot e^{k^2} \cdot e^{k^3} \dots = e^3 \Rightarrow e^{1+k+k^2+k^3+\dots} = e^3$

$$\Leftrightarrow 1 + k + k^2 + k^3 + \dots = 3 \Rightarrow G_1 = 1, r = k$$

$$\Rightarrow \frac{1}{1-k} = 3 \Rightarrow 1 = 3 - 3k \Rightarrow -3k = -2 \Rightarrow k = \frac{2}{3}$$

Find the value of m : if

a) $\sum_{k=0}^{\infty} e^{mk} = 9$

b) $\sum_{k=1}^{\infty} 6^{mk} = \frac{1}{5}$

Solution: Here

a) $\sum_{k=0}^{\infty} e^{mk} = 9 \Rightarrow 1 + e^m + e^{2m} + e^{3m} + \dots = 9$ the series

converges to the sum 9. $G_1 = 1, r = e^m$

$$\Rightarrow \frac{G_1}{1-r} \Rightarrow \frac{1}{1-e^m} = 9 \Rightarrow 1 = 9 - 9e^m \Rightarrow -9e^m = -8$$

$$\Rightarrow e^m = \frac{8}{9} \Rightarrow m = \log_e^8 = \ln \frac{8}{9} = \ln 8 - \ln 9$$

$$\therefore m = \ln \frac{8}{9} = \ln 8 - \ln 9$$

b) $\sum_{k=1}^{\infty} 6^{mk} = 6^m + 6^{2m} + 6^{3m} + \dots = \frac{1}{5} \Rightarrow G_1 = 6^m, r = 6^m$

$$\Rightarrow S_{\infty} = \frac{G_1}{1-r}$$

$$\Rightarrow \frac{6^m}{1-r} \Rightarrow \frac{1}{5} \Rightarrow (5)6^m = 1 - 6^m \Rightarrow 5(6)^m + 6^m = 1$$

$$\Rightarrow 6^m(5+1) = 1 \Rightarrow 6^m = \frac{1}{6} \Leftrightarrow 6^m = 6^{-1} \Leftrightarrow m = -1$$

75. Solve a) $\sum_{n=1}^{\infty} x^n = \frac{3}{8}$

b) $\sum_{n=0}^{\infty} y^n = x$

Solution: a) $\sum_{n=1}^{\infty} x^{2n-1} = x + x^3 + x^5 + \dots = \frac{3}{8}, G_1 = x, r = x^2$

$$\Rightarrow \frac{G_1}{1-r} \Rightarrow \frac{x}{1-x^2} = \frac{3}{8} \Rightarrow 8x = 3 - 3x^2 \Rightarrow 3x^2 + 8x - 3 = 0$$

$$\Rightarrow x = \frac{1}{3}, x = -3 \text{ we take only } x = \frac{1}{3}$$

$$b) \sum_{n=1}^{\infty} y^n = 1 + y + y^2 + \dots = x, G_1 = 1, r = y$$

$$\Rightarrow S_{\infty} = \frac{G_1}{1-r} \Rightarrow x = \frac{1}{1-y} \Rightarrow x - xy = 1$$

$$\Rightarrow x - 1 = xy \Rightarrow y = \frac{x-1}{x}, \text{ for } x \neq 0$$

76. Find the radius of convergence and sum of the series

$$a) \sum_{k=0}^{\infty} (x-2)^k$$

$$b) \sum_{k=1}^{\infty} (2x+1)^k$$

Solution:

$$a) \sum_{k=0}^{\infty} (x-2)^k = 1 + (x-2) + (x-2)^2 + (x-2)^3 + \dots$$

$$G_1 = 1, r = x - 2$$

$$\Rightarrow \text{Convergent if } |r| < 1 \Rightarrow |x-2| < 1$$

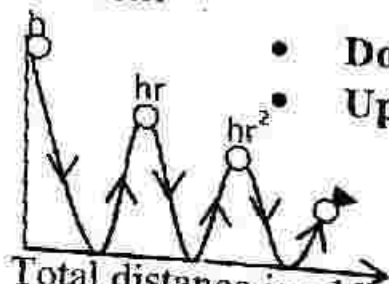
$$\Rightarrow -1 < x-2 < 1 \Rightarrow 1 < x < 3 \leftarrow \text{radius convergent}$$

$$\Rightarrow \text{sum, } S_{\infty} = \frac{G_1}{1-r} = \frac{1}{1-(x-2)} = \frac{1}{3-x}$$

A bouncing Ball problem

A ball dropped from a height of h meter. In each bounce, it rebounds r of the height from which it falls. Find the total distance the ball travel up and down.

Solution:



$$\bullet \text{ Down ward series: } h + hr + hr^2 + hr^3 + \dots + hr^{n-1} + \dots$$

$$\bullet \text{ Upward series: } hr + hr^2 + hr^3 + \dots + hr^n + \dots$$

Total distance is adding down ward and upward series:

$$\Rightarrow S = h + 2hr + 2hr^2 + 2hr^3 + \dots + 2hr^n + \dots$$

$$S = h + \frac{2hr}{1-r} = h \left[\frac{1+r}{1-r} \right] \leftarrow \text{the total vertical distance}$$

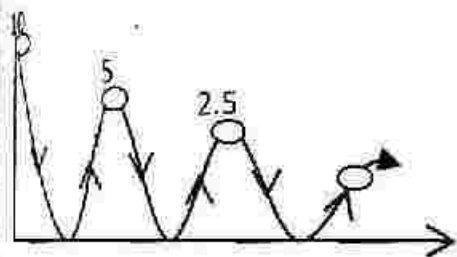
Illustrative example

A ball is dropped from height of 16 meter. Each time it drops h meter, it rebounds $0.81h$ meters. Find the total vertical distance traveled by the ball.

Solution: Total distance traveled $= h \left[\frac{1+r}{1-r} \right]$

$$h=16, r=0.81 \Rightarrow S = 16 \left[\frac{1+0.81}{1-0.81} \right] = 16 \left(\frac{1.81}{0.19} \right) = 152.42m$$

A ball is dropped from height of 10 meter. It rebounds one-half the distance after each fall.



a) How far does it bound up on the seventh rebounds.

b) Find the total distance the ball

has traveled when it hits the ground the sixth time.

c) Find the total (maximum) distance the ball travels before comes to rest.

Solution: a) $G_n = Gr^{n-1} = hr^{n-1}$, $h = 10$, $r = \frac{1}{2}$

$$\Rightarrow G_5 = (10) \left(\frac{1}{2} \right)^{5-1} = (10) \left(\frac{1}{2} \right)^4 = \frac{10}{16} = \frac{5}{8}m$$

$$S_6 = h + 2hr^5$$

b) Total distance the ball traveled when hits the ground the sixth time $\Rightarrow S_6 = h + 2hr + 2hr^2 + 2hr^3 + 2hr^4 + 2hr^5$

$$= 10 + 2(10) \left(\frac{1}{2} \right) + 2(10) \left(\frac{1}{4} \right) + 2(10) \left(\frac{1}{8} \right) + 2(10) \left(\frac{1}{16} \right) + 2(10) \left(\frac{1}{32} \right)$$

$$= 10 + 10 + 5 + \frac{10}{4} + \frac{10}{8} + \frac{10}{16} + \frac{10}{32} \approx 29m$$

c) Total distance before comes to rest is

$$\Rightarrow S = h \left[\frac{1+r}{1-r} \right] = 10 \left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right) = 10(3) = 30m$$

79. A ball is dropped from height of 5 meter the ball rebounds $\frac{60}{100}$ of the distance after each fall. Find the total distance.

Solution: $h = 5m$, $r = 60\% = 0.6$

$$\text{Total distance } S = 5 \left[\frac{1+0.6}{1-0.6} \right] = \frac{5(1.6)}{0.4} = 20m$$

Solved Problem

78. Let $a_n = n + \cos n\pi$, $n = 1, 2, 3, \dots$ be the n^{th} term of sequence. Then which of the following is true...

a) $a_5 = 6$ b) $a_{10} = 9$ c) $a_7 + a_8 = 15$ d) $a_7 - a_8 = 1$

Solution: a) $a_5 = 5 + \cos(5\pi) = 5 + (-1) = 4$

b) $a_{10} = 10 + \cos(10\pi) = 10 + 1 = 11$

c) $a_7 + a_8 = 7 + \cos(7\pi) + (8 + \cos 8\pi) = 7 + (-1) + 8 + (1) = 15$

d) $a_7 - a_8 = 7 + (-1) - (6 + 1) = -1$

Answer: C

79. If $\{A_n\}$ is an arithmetic sequence with $A_1 = 5$ and $A_5 = 21$, then

the partial sum $\sum_{n=1}^{20} A_n$ is equal to

A. 760 B. 780 C. 860 D. 870

Solution: $d = \frac{A_5 - A_1}{4} = \frac{21 - 5}{4} = 4$

$$\sum_{n=1}^n A_n \Rightarrow S_n = \frac{n}{2} (2A_1 + (n-1)d) \Rightarrow S_{20} = \frac{20}{2} (2(5) + 19(4)) = 860$$

80. What is the sum of the series $\sum_{n=0}^{\infty} 2^n 3^{2-n}$?

A. 6 B. 18 C. 27 D. ∞

Solution:

$$\sum_{n=0}^{\infty} 2^n 3^{2-n} = \sum_{n=0}^{\infty} 2^n \cdot 3^2 \cdot 3^{-n} = \sum_{n=0}^{\infty} 9 \left(\frac{2}{3} \right)^n, G_1 = 9, r = \frac{2}{3}$$

$$\Rightarrow S_{\infty} = \frac{G_1}{1-r} = \frac{9}{1-\frac{2}{3}} = 9 \times 3 = 27$$

Answer: C

Find the sum of all integers between 30 and 201 which are

- a. multiple of 4 b. multiple of 7
c. multiple of 4 and 7 d. multiple of 4 or 7

Solution: $S_n = \frac{n}{2}[A_1 + A_n]$ and $n = \frac{A_n - A_1}{d} + 1$

a. $n = \frac{200 - 32}{4} + 1 = 43$ and $S_{43} = [32 + 200] = 4644$

b. $n = \frac{196 - 35}{7} + 1 = 24$ and $S_{24} = \frac{24}{2}[35 + 196] = 2772$

c. Integer multiple of 4 and 7, must be divisible by LCM of 4 and 7 i.e 28 So, $A_1 = 56$, $d = 28$, $A_n = 196$

$$\Rightarrow n = \frac{196 - 56}{28} + 1 = 6$$

$$\therefore S_6 = \frac{6}{2}[56 + 196] = 756$$

d. Sum of integer multiple of 4 or 7.

$$= \text{sum of } [(\text{multiple of 4} + \text{multiple of 7}) - (\text{multiple of 4 and 7})]$$

$$= 4644 + 2772 - 756 = 6660$$

A ball is thrown vertically from the ground up to a height of 20 meter. Each time it drops h meters, it rebounds $\frac{3}{4}$ of the height.

What is the total vertical distance travelled by the ball before it comes to rest?

- A. 80m B. 160m C. 120m D. 60m

Solution: $G_1 = h = 20\text{m}$, $r = \frac{3}{4}$

$$\Rightarrow S_{\infty} = \frac{2G}{1-r} = \frac{2h}{1-r} = \frac{2(20)}{1-\frac{3}{4}} = 160\text{m}$$

Suppose $\{a_n\}$ is an A.P such that $a_5 = 21$ and $a_{10} = 66$ then a_2 is equal to:..... EHEECE

- A. -6 B. -15 C. 3 D. 9

$$\text{Solution: } d = \frac{a_{10} - a_5}{10 - 5} = \frac{66 - 21}{5} = \frac{45}{5} = 9$$

$$\text{Also } \frac{a_5 - a_2}{5 - 2} = 9 \Rightarrow \frac{21 - a_2}{3} = 9 \Rightarrow 21 - a_2 = 27 \Rightarrow a_2 = -6$$

84. $\sum_{n=1}^{\infty} \left[\left(-\frac{1}{3} \right)^n + 4^{1-n} \right]$ equal

A. $\frac{1}{12}$ B. $\frac{19}{12}$ C. $\frac{13}{12}$ D. $\frac{25}{12}$

$$\begin{aligned} \text{Solution: } \sum_{n=1}^{\infty} \left[\left(-\frac{1}{3} \right)^n + 4^{1-n} \right] &= \sum_{n=1}^{\infty} \left(-\frac{1}{3} \right)^n + \sum_{n=1}^{\infty} (4) \left(\frac{1}{4^n} \right) \\ &= \frac{1}{1 + \frac{1}{3}} + \frac{1}{1 - \frac{1}{4}} = \frac{13}{12} \end{aligned}$$

85. If k^{th} term of a sequence is $a_k = \frac{(-1)^k (2-k)}{2k-1}$, then next term a_{k+1} when k is even.

A. $\frac{k-1}{2k+1}$ B. $\frac{3-k}{2k+3}$ C. $\frac{k-3}{2k+1}$ D. $\frac{1-k}{2k-1}$

Solution:

$$a_{k+1} = \frac{(-1)^{k+1} (2 - (k+1))}{2(k+1) - 1} = \frac{(-1)^k \cdot (-1)(1-k)}{2k+1} = \frac{k-1}{2k+1}$$

86. The sequence $\{a_n\}$, $n \geq 1$, where $a_1 = 2$ and $a_{n+1} = a_n + 4$

Then $a_3 + a_4 + a_5 + \dots + a_{35}$ is equal to EHEECE

A. 2460 B. 2458 C. 2442 D. 2450

$$\text{Solution: } d = \frac{a_{n+1} - a_n}{n+1 - n} = \frac{a_n + 4 - a_n}{1} = 4$$

$$\Rightarrow \text{Total term from 3 to 35 is} = 35 - 3 + 1 = 33$$

$$\Rightarrow S_{33} = \frac{33}{2} (2A_1 + 32d) = \frac{33}{2} (2(10) + 32(4)) = 2442$$

Explanation
 $a_3 = A_1 = 10$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{(3^n)^2}$ is equal to EHEECE

- A. $\frac{9}{8}$ B. $\frac{9}{10}$ C. $\frac{-9}{10}$ D. $\frac{-1}{10}$

Solution: $\sum_{n=1}^{\infty} \frac{(-1)^n}{(3^n)^2} = \frac{-1}{3^2} + \frac{1}{3^4} + \frac{-1}{3^6} + \dots$ $G_1 = \frac{-1}{9}$, $r = -\frac{1}{9}$

$$\Rightarrow S_{\infty} = \frac{G_1}{1-r} = \frac{\frac{-1}{9}}{1 + \frac{1}{9}} = \frac{-1}{9} \cdot \frac{9}{10} = -\frac{1}{10}$$

Answer: D

18. Let $S_n = 4n^2$, $n = 1, 2, 3, \dots$ be the sum of the first n terms a sequence $\{a_n\}_{n=1}^{\infty}$. Then a_3 UEE

- A. 8 B. 12 C. 16 D. 20

Solution:

$$a_n = S_n - S_{n-1} \Rightarrow a_3 = S_3 - S_2 = 4(3)^2 - 4(2)^2 = 36 - 16 = 20$$

19. Let a sequence $\{a_n\}_{n \geq 1}$ with $a_1 = 2$, and $a_n = 3a_{n-1}$ for all $n \geq 2$.

Then $\sum_{i=1}^{20} a_i$ is equal to UEE

- A. $3^{20} - 1$ B. $1 - \frac{1}{3^{20}}$ C. $1 - \frac{1}{2^{21}}$ D. $3^{20} - 1$

Solution: $a_1 = 2$, $a_2 = 3a_1 = 3(2) = 6$, $a_3 = 3a_2 = 3(6) = 18$
 $\Rightarrow 2, 6, 18, 54, \dots \Rightarrow G_1 = 2, r = 3$. It is G.P

$$\Rightarrow \sum_{i=1}^{20} a_i \Rightarrow S_{20} = G_1 \left[\frac{r^n - 1}{r - 1} \right] = 2 \left[\frac{3^{20} - 1}{3 - 1} \right] = 3^{20} - 1$$

Answer: A

20. Let $\{a_n\}_{n \geq 1}$ be sequence with $a_1 = 3$ and $a_{n+1} = na_n$ then $a_n =$ _____

- A. $3n!$ B. $3(n-1)!$ C. $(3n)!$ D. $3(n+1)!$

Solution: $a_{n+1} = na_n \Rightarrow a_{1+1} = 1a_1 = 3$, $a_2 = 3(1)$

$\Rightarrow n=2 \Rightarrow a_{2+1} = 2a_2 = 2(3) = 6 \Rightarrow a_3 = 6 = 3 \times 2 \times 1 = 3(2 \times 1)$

$\Rightarrow n=3 \Rightarrow a_{3+1} = 3a_3 = 3(6) = 18$

$\Rightarrow a_4 = 18 = 3(3 \times 2 \times 1) = 3(3 \times 2 \times 1)$

$3(3!)$

$$\Rightarrow a_5 = 72 = 3(4 \times 3 \times 2 \times 1) \Rightarrow a_n = 3(n-1)! \quad n \geq 1$$

91. Let $\{a_n\}$ be A.P 20, 16, 12, 8, ... Then $a_{200} - a_{203}$
 A. -12 B. 8 C. 12 D. 16

Answer: B

$$\text{Solution: } d = \frac{a_m - a_k}{m - k} \Rightarrow \frac{a_{200} - a_{203}}{200 - 203} = -4$$

$$\Rightarrow \frac{a_{200} - a_{203}}{-3} = -4 \Rightarrow a_{200} - a_{203} = 12$$

Answer: C

92. If $a_n = [3^{-1/n} \sqrt[n]{5}]^n$ for $n = 1, 2, 3$ then the sum of the series $a_1 + a_2 + a_3 + \dots + a_n + \dots$ is

- A. $\frac{5}{3}$ B. $\frac{5}{2}$ C. $\frac{15}{2}$ D. $\frac{10}{9}$

$$\text{Solution: } a_n = [3^{-1/n} \sqrt[n]{5}]^n \Rightarrow [3^{-1} \cdot 5^1]^n = 3^{-n} (5) = 5(3^{-n})$$

$$\Rightarrow a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} 5(3^{-1}) = \sum_{n=1}^{\infty} \frac{5}{3^n}$$

$$\Rightarrow S_{\infty} = \frac{G_1}{1-r} = \frac{\frac{5}{3}}{1-\frac{1}{3}} = \frac{5}{3} \cdot \frac{3}{2} = \frac{5}{2}$$

93. $\sum_{n=6}^{\infty} 8(0.5)^n$ is equal to

- A. $\frac{1}{2}$ B. $\frac{1}{4}$ C. 8 D. 16

$$\text{Solution: } \sum_{n=m}^{\infty} ar^n = \frac{G_1}{1-r} = \frac{ar^m}{1-r} = \frac{8\left(\frac{1}{2}\right)^6}{1-\frac{1}{2}} = \frac{1}{8} (2) = \frac{1}{4}$$

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Let $\{A_n\}$ be an arithmetic sequence with $A_{12} = 16$ and $A_{23} = 49$, then find a) d b) n^{th} term

$$\text{Solution: } d = \frac{A_{23} - A_{12}}{23 - 12} = \frac{49 - 16}{23 - 12} = \frac{33}{11} = 3 \leftarrow \text{common}$$

difference

$$\text{b) } \frac{A_n - A_{12}}{n - 12} = 3 \Rightarrow A_n - A_{12} = 3(n - 12) \Rightarrow A_n = 3n - 36 + 16$$

$$\Rightarrow A_n = 3n - 20 \leftarrow n^{\text{th}} \text{ term}$$

Binomial series

Expansion of $(a + x)^n$

A for positive integral values of n .

$$(a+x)^n = a^n + (n)a^{n-1}x + \frac{n(n-1)a^{n-2}x^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}x^3}{3!} + \dots + x^n$$

B) For Negative and fractional values of n

The binomial theorem holds also for negative and fractional values of n for $(a+x)^n$, where x is numerically less than a . However when n is a negative or fractional number the expansion does not terminate.

Illustrative example

$$1) (a+x)^3 = a^3 + 3a^2x + \frac{(3)(2)ax^2}{2} + \frac{(3)(2)(1)x^3}{3!}$$

2) Write the first four terms of the expansion.

$$\text{a) } (a+x)^{-1} = a^{-1} + (-1)a^{-2}x + \frac{(-1)(-2)a^{-3}x^2}{2!} + \frac{(-1)(-2)(-3)a^{-4}x^3}{3!} + \dots$$

for $|x| < |a|$

$$\text{b) } (1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)(-x)^2}{(2)(1)} + \frac{(-2)(-3)(-4)(-x)^3}{3 \times 2 \times 1} + \dots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots \text{ valid for } |x| < 1$$

$$\text{c) } (1-x)^{-4} = 1 + (-4)(-x) + \frac{(-4)(-5)x^2}{2 \times 1} + \frac{(-4)(-5)(-6)}{3 \times 2 \times 1}x^3 + \dots$$

$$= 1 + 4x + 10x^2 + 20x^3 + \dots$$

$$\text{d) } (2+x)^{-3} = 2^{-3} + (-3)(2)^{-4}x + \frac{(-3)(-4)2^{-5}x^2}{2 \times 1} + \frac{(-3)(-4)(-5)(2^{-6})x^3}{3 \times 2 \times 1} + \dots$$

$$= \frac{1}{8} - \frac{3}{16}x + \frac{3}{16}x^2 - \frac{5}{32}x^3 + \dots \text{ valid for } |x| < 2$$

$$\begin{aligned} \text{e) } (1+x)^{\frac{3}{2}} &= 1 + \frac{3}{2}x + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)x^2}{2 \times 1} + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)x^3}{3 \times 2 \times 1} + \dots \\ &= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \dots \quad |x| < 1 \end{aligned}$$

$$\text{f) } \frac{1}{\sqrt{4-x}} \quad \text{Note } \frac{1}{\sqrt{4-x}} = \left(\sqrt{4-x}\right)^{-1} = (4-x)^{-\frac{1}{2}}$$

$$\begin{aligned} \therefore (4-x)^{-\frac{1}{2}} &= 4^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)4^{-\frac{3}{2}}(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)4^{-\frac{5}{2}}(-x)^2}{2 \times 1} \\ &\quad + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)4^{-\frac{7}{2}}(-x)^3}{3 \times 2 \times 1} + \dots \\ &= \frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2 + \dots \end{aligned}$$

3) Write the sixth term of the expansion of a) $(1-x^2)^{\frac{1}{2}}$

Solution: k^{th} term of $(a+x)^n$

$$= \frac{n(n-1)(n-2)\dots(n-k+2)a^{n-k+1}x^{k-1}}{(k-1)!}$$

$$\therefore n = \frac{1}{2}, k = 6, n-k+2 = \frac{1}{2} - 6 + 2 = -\frac{7}{2}, k-1 = 5, \text{ and}$$

$$n-k+1 = \frac{1}{2} - 6 + 1 = -\frac{9}{2}$$

$$\therefore 6^{\text{th}} \text{ term} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{1 \times 2 \times 3 \times 4 \times 5} (-x^2)^5 = \frac{-7}{256}x^9$$

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The 4th term of the expansion $(a)(1+x)^{\frac{1}{3}}$

Solution: $n = \frac{1}{3}$, $k = 4$, $n - k + 2 = \frac{1}{3} - 4 + 2 = -\frac{5}{3}$

And $n - k + 1 = \frac{1}{3} - 4 + 1 = -\frac{8}{3}$, $k - 1 = 3$

$$\therefore 4^{\text{th}} \text{ term} = \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(1)^{-\frac{8}{3}}}{1 \times 2 \times 3} \cdot x^3 = \frac{5}{81}x^3$$

b) 4th term of $(a+x)^{-\frac{1}{2}}$

Exercise for the reader

5) Evaluate the following using binomial series

a) $\sqrt{5}$

b) $\sqrt{26}$

c) $\sqrt[3]{9}$

Solution:

$$\begin{aligned} \text{a) } \sqrt{5} &= (4+1)^{\frac{1}{2}} = 4^{\frac{1}{2}} + \left(\frac{1}{2}\right)(4)^{-\frac{1}{2}}(1) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(4)^{-\frac{3}{2}}}{2 \times 1} + \dots \\ &= 2 + \frac{1}{4} + -\frac{1}{64} + \dots \approx 2.24 \end{aligned}$$

$$\begin{aligned} \text{b) } \sqrt{26} &= (25+1)^{\frac{1}{2}} = (25)^{\frac{1}{2}} + \frac{1}{2}(25)^{-\frac{1}{2}} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(25)^{-\frac{3}{2}} + \dots \\ &= 5 + \frac{1}{10} + -\frac{1}{1000} + \frac{2}{10^5} \dots = 5.09902 \end{aligned}$$

$$\begin{aligned} \text{c) } \sqrt[3]{9} &= (8+1)^{\frac{1}{3}} = 8^{\frac{1}{3}} + \frac{1}{3}8^{-\frac{2}{3}} + \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)8^{-\frac{5}{3}} + \dots \\ &= 2 + \frac{1}{12} - \frac{1}{144} + \dots \end{aligned}$$

Question for Concept

1. Discuss the difference between a finite sequence and an infinite sequence.
2. Discuss the difference between a sequence and series
3. What is a recursive sequence?
4. What is an alternating sequence. Give an example

Supplementary Exercise on Unit – 10

1. If $A_5 = 8$ and $d = 3$, then find:
a. A_1 b. A_{20} c. n th term (A_n) d. A_{n+1}
2. If $A_1 = 11$ and $A_8 = 13$ then find
a. Common difference b. A_1 c. n th term
3. The k^{th} term of an A.P is $2m + 1$ and the m^{th} term is $2k + 1$. Find $(k + m)^{\text{th}}$ term.
4. Find the total number of natural number between 100 and 1000 divisible by six.
5. In each part, find the common difference of A_n :
a. $A_n = 3n + 1$ b. $A_n = \frac{2 - 7n}{3}$
6. Let $\{a_n\}$ be an arithmetic sequence: 28, 25, 22, ... Then find
a. $a_{307} - a_{309}$ b. $a_{200} - a_{204}$
7. Find a common difference in terms of n if n arithmetic means are inserted:
a. Between 3 and 7 c. between m and k
b. Between 2 and 3
8. If $G_1 = 8$, $r = \frac{1}{2}$, then find
a. G_6 b. G_8 c. G_n
9. If $G_4 = 3$ and $G_7 = 81$ then find:
a. Common ratio (r) b. G_1 c. G_n
10. If $G_3 = 1$ and $G_6 = 64$, then find:
a. r b. G_1 c. G_n
11. $G_2 = \frac{1}{\sqrt{2}}$, $G_7 = -\frac{1}{8}$, find r , G_1 , G_5 and G_n .
12. If x , 9, y are in geometric progression and x , 15, y are in arithmetic progression then find the value of x and y .
13. If $x + 1$, $-5x + 4$, and $2x + 8$ are consecutive G.P, find the value of x .

Find the sum (k^{th} partial sum).

a. $\sum_{k=1}^{15} 6$

b. $\sum_{k=4}^{21} 9$

c. $\sum_{k=12}^{100} 7$

d. $\sum_{k=1}^{10} 2k$

e. $\sum_{k=1}^{10} k^3$

f. $\sum_{k=1}^{13} \left(\frac{2}{n} - \frac{2}{n+1} \right)$

If $A_6 = 23$ and $A_{15} = 50$, then find:

a. S_{10} b. S_{20} c. $\sum_{n=1}^{24} A_n$

d. $\sum_{n=1}^{12} A_n$

Find the indicated term of an A. P:

a. If $S_8 = 96$, $A_1 = 5$, then find A_8 , A_4 , and S_{10}

b. $S_6 = 60$, $A_8 = 19$, then find A_1 , A_{10} and A_n

If $S_4 = 40$ and $S_{10} = 160$, then find:

d. b. nth term c. $\sum_{n=1}^n A_n$ d. $\sum_{n=1}^{20} A_n$

Find the n^{th} partial sum (S_n) for the given n^{th} term arithmetic sequences.

a. $A_n = 3n + 1$ b. $A_n = 2n + 3$ c. $a_n = 2 - 5n$

Find the n^{th} term (A_n) and common difference for the given n^{th} partial sum (S_n) of arithmetic sequence.

a. $S_n = 3n^2 + 2nb$. $S_n = \frac{1}{2}(Sn^2 - n)$

a. If $A_4 = 11$, $A_7 = 20$ of an A. P then find $\sum_{n=1}^{20} A_n$

b. If $A_5 = 4$, $A_{12} = 18$ of an A.P then find $\sum_{n=1}^{30} A_n$

The sequence $\{a_n\}$, $n \geq 1$ where $a_1 = 3$ and $a_{n+1} = a_n + 2$ then $a_4 + a_5 + a_6 + \dots + a_{29}$ is equal to _____

Find the sum of all number from 100 to 300.

a. Which are exactly divisible by 8.

b. Which are exactly divisible by 7.

The sum of the first 10 terms of an A.P is 210 and the sum of the first 20 terms is 820. Find the sum of the first 30 terms.

24. The sum of the first 8 terms of an A.P is 104 and the sum of the next (second) 12 terms is 636. Find the general term, common difference.
25. a. Find the number of integers between 32 and 395 that are divisible by 6.
b. Find the number of negative integers greater than -590 that are divisible by 33.
26. Find the sum
- a. $\sum_{k=1}^{20} (3k-5)$ b. $\sum_{k=10}^{28} (2k+3)$ c. $\sum_{k=1}^{12} (4k-1)$
- d. $\sum_{n=1}^{99} \log \frac{n+1}{n}$ e. $\sum_{k=1}^{100} \frac{1}{(k+1)(k+2)}$ f. $\sum_{k=0}^{60} x^3$
27. Find the sum of all 3 digit whole numbers that are divisible by
- a. 11 b. 13 c. 24 d. 31
28. Find the sum of:
- a. 5 arithmetic mean between 3 and 7
b. 10 arithmetic mean between 1 and 2
c. n arithmetic mean between 2 and 3
d. n arithmetic mean between m and k
29. Find the sum of 16 terms of the A.P -10, -8, -16, -4, ... starting from the 10th term.
30. Find the sum of whole numbers that are:
- a. Less than 200 and leave remainder 3 when divided by 5.
b. Between 100 and 300 leave remainder 2 when divided by 5.
31. If $G_4 = 24$, and $G_7 = 192$, then find the sum for the first indicated terms.
- a. S_6 b. S_8 c. S_{10} d. S_n
32. Find the partial sum of each geometric sequence.
- a. $\sum_{k=1}^7 3(2^{k-1})$ b. $\sum_{k=2}^8 32\left(\frac{1}{3}\right)^k$ c. $\sum_{k=1}^{10} \left(\frac{3}{2}\right)^k$
33. If $G_4 = 4$, and $G_6 = 16$, then find ($r > 0$).
- a. r b. S_6 c. $\sum_{n=1}^n G_n$

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Find the sum of infinite geometric series.

a. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$

c. $\frac{1}{5} - \frac{2}{5^2} + \frac{2^2}{5^3} - \frac{2^3}{5^4} + \dots$

b. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots$

d. $\frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{9} + \frac{1}{32} + \frac{1}{27} + \dots$

Find the sum if exists

a. $\sum_{n=0}^{\infty} (0.7)^n$

b. $\sum_{n=2}^{\infty} (0.7)^n$

c. $\sum_{n=0}^{\infty} (-0.6)^n$

d. $\sum_{n=1}^{\infty} 20(0.9)^{n-1}$

e. $\sum_{n=1}^{\infty} 10(0.25)^{n-1}$

f. $\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1}$

g. $\sum_{k=2}^{\infty} (5^{4-k})$

h. $\sum_{k=2}^{\infty} \left(-\frac{3}{2}\right)^{1-k}$

i. $\sum_{k=1}^{\infty} 6\left(\frac{4}{3}\right)^{2-k}$

j. $\sum_{k=1}^{\infty} \frac{3^{k+2}}{4^{k-1}}$

k. $\sum_{k=1}^{\infty} (5^k)(7^{1-2k})$

l. $\sum_{k=0}^{\infty} \left(\frac{2}{3^{k-1}} - \frac{3}{4^k}\right)$

Find the product of each of the following expression.

a. $3.3^{\frac{1}{3}}.3^{\frac{1}{9}}.3^{\frac{1}{27}} \dots$

b. $4.4^{\frac{1}{5}}.4^{\frac{1}{25}}.4^{\frac{1}{125}} \dots$

Suppose: $6^k \cdot 6^{k^2} \cdot 6^{k^3} \dots = 36$, then find the value of k.

Find the value (s) of k if

a. $\sum_{m=1}^{\infty} 4^{mk} = \frac{1}{15}$

b. $\sum_{m=0}^{\infty} 3^{mk} = \frac{27}{26}$

c. $\sum_{m=0}^{\infty} \left(\frac{3}{2}\right)^{mk} = \frac{9}{5}$



Introduction to limit and continuity

Lower and Upper bounds of sequence

Definition: Let $\{a_n\}_{n=1}^{\infty}$ be sequence of number Then:

- i) A number u is called an upper bounds of $\{a_n\}_{n=1}^{\infty}$ if u is greater than or equal to every number in $\{a_n\}_{n=1}^{\infty}$ i.e., $\forall a_i \in \{a_n\} \Rightarrow u \geq a_i$.

The smallest of all upper bounds of $\{a_n\}_{n=1}^{\infty}$ is called least upper bounds (**lub**) of $\{a_n\}_{n=1}^{\infty}$.

- ii) A number ℓ is called a lower bounds of $\{a_n\}_{n=1}^{\infty}$ if ℓ is smaller than or equal to every number in $\{a_n\}_{n=1}^{\infty}$ i.e., for all $a_i \in \{a_n\}_{n=1}^{\infty} \Rightarrow \ell \leq a_i$. The largest of all lower bounds of $\{a_n\}_{n=1}^{\infty}$ is called greatest lower bound (**glb**) of $\{a_n\}_{n=1}^{\infty}$.

- iii) A sequence of $\{a_n\}_{n=1}^{\infty}$ is said to be bounded if it has an upper bound (bounded above) and lower bound (bounded below).

Illustrative Example

1. Find: i) The set of all upper bound
ii) Smallest of all upper bound (Lub)
iii) the set of all lower bound
iv) the greatest of all lower bounds (glb)

a. $\{2, 3, 4, 5\}$

b. $(1, 3]$

c. $\left\{\frac{1}{n}\right\}$

d. $\left\{\frac{2n+1}{n}\right\}$

e. $\left\{\frac{2n-4n}{3n+6}\right\}$

f. $\left\{(-1)^n + \frac{1}{n+1}\right\}$

- f. $\left\{ \frac{n-2}{n+2} \right\}$ h. $\left\{ \frac{n+4}{5n-6} \right\}$ i. $\left\{ n^{\frac{1}{n}} \right\}$
 g. $\{ \ln(n) \}$ k. $\{ \cos n^2 \}$ l. $\{ \sin^2(n) \}$
 m. $\left\{ \frac{3}{n!} \right\}$ n. $\left\{ 3^n (-1)^n \right\}$ o. $\left\{ \frac{n!}{n^n} \right\}$

Solution:

a. i. the set of all upper bounds is $[5, \infty)$

ii. the smallest of all upper bounds (lub) is 5

iii. the set of all lower bounds are $(-\infty, 2]$

iv. the greatest of all lower bounds is 2

b. i. $[3, \infty) \leftarrow$ the set of all upper bound of $(1, 3)$

ii. $3 \leftarrow$ lub (least upper bound)

iii. the set of all lower bounds of $(1, 3]$ are $(-\infty, 1]$

iv. the greatest of all lower bound glb is 1

c. The terms of the sequence $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$ have the pattern

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, 0 \Rightarrow 0 < \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \leq 1 \text{ thus}$$

i. the set of all upper bound are $[1, \infty)$

ii. the smallest of all upper bounds: lub is 1

iii. the set of all lower bounds are $(-\infty, 0]$

iv. glb (greatest lower bound) is 0

$$d. \left\{ \frac{2n+1}{n} \right\} = \left\{ 2 + \frac{1}{n} \right\}_{n=1}^{\infty} = \left\{ 3, 2 + \frac{1}{2}, 2 + \frac{1}{3}, 2 + \frac{1}{4}, \dots, 2 \right\}$$

$$\text{decreasing} \Rightarrow 2 < \left\{ \frac{2n+1}{n} \right\}_{n=1}^{\infty} \leq 3, \text{ thus}$$

i. $[3, \infty) \leftarrow$ all upper bound

ii. $\text{lub} = 3 \leftarrow$ the smallest of all upper bound

iii. $(-\infty, 2] \leftarrow$ all lower bound

iv. $\text{glb} = 2 \leftarrow$ the largest of all lower bound

e. $\left\{ \frac{2-4n}{3n+6} \right\}_{n=1}^{\infty}$ the numerator and denominator divide by n , the same

as $\left\{ \frac{\frac{2}{n} - \frac{4n}{n}}{\frac{3n}{n} + \frac{6}{n}} \right\}_{n=1}^{\infty} = \left\{ \frac{\frac{2}{n} - 4}{3 + \frac{6}{n}} \right\}_{n=1}^{\infty}$ as $n \rightarrow \infty$ (very large) $\frac{2}{n} \rightarrow 0$ and

$\frac{6}{n} \rightarrow 0 \Rightarrow \Rightarrow \frac{-4}{3} < \left\{ \frac{2-4n}{3n+6} \right\} \leq \frac{-2}{9} \leftarrow$ decreasing to $\frac{-4}{3}$ as $n \rightarrow \infty$

As n increase from $1 \rightarrow \infty$, the sequence decrease from

$\frac{-2}{9}, -\frac{1}{2}, \frac{-2}{3}, \dots, \frac{-4}{3}$, thus:

i. $\left[-\frac{2}{9}, \infty \right) \leftarrow$ The set of all upper bound

ii. $\text{lub} = -\frac{2}{9} \leftarrow$ smallest of all upper bound

iii. $\left(-\infty, -\frac{4}{3} \right) \leftarrow$ The set of all lower bound

iv. $\text{glb} = -\frac{4}{3} \leftarrow$ the largest of all lower bound

f. For odd number $n = 1, 3, 5, 7, \dots, \infty$ then

$$\left\{ (-1)^n + \frac{1}{n+1} \right\}_{n=1}^{\infty} = \left\{ -1 + \frac{1}{n+1} \right\}_{n=1}^{\infty} = \left\{ -\frac{1}{2}, -\frac{3}{4}, -\frac{5}{6}, \dots, \frac{-n}{n+1}, -1 \right\}$$

For even number $n = 2, 4, 6, \dots, \infty$ then

$$\left\{ (-1)^n + \frac{1}{n+1} \right\}_{n=1}^{\infty} = \left\{ 1 + \frac{1}{n+1} \right\}_{n=1}^{\infty} = \left\{ \frac{n+2}{n+1} \right\}_{n=1}^{\infty} = \left\{ \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \dots, 1 \right\} \text{ so that, } -1 <$$

$$\left\{ (-1)^n + \frac{1}{n+1} \right\}_{n=1}^{\infty} \leq \frac{4}{3}, \text{ thus}$$

i. all upper bounds are $\left[\frac{4}{3}, \infty\right)$

ii. $\text{lub} = \frac{4}{3}$

iii. set of all lower bounds are $(-\infty, -1]$

iv. $\text{glb} = -1$

$$\left\{\frac{n-2}{n+2}\right\}_{n=1}^{\infty} = \left\{-\frac{1}{3}, 0, \frac{1}{5}, \frac{2}{6}, \frac{3}{7}, \dots, 1\right\} \text{ so that } -\frac{1}{3} \leq \left\{\frac{n-2}{n+2}\right\}_{n=1}^{\infty} < 1,$$

thus i. set of all upper bound $[1, \infty)$

ii. $\text{lub} = 1$

iii. set of all lower bound $\left(-\infty, -\frac{1}{3}\right)$

iv. $\text{glb} = -\frac{1}{3}$

$$-5 \leq \left\{\frac{n+4}{5n-6}\right\}_{n=1}^{\infty} < \frac{1}{5} \leftarrow \text{exercise left for you}$$

$$\left\{n^{\frac{1}{n}}\right\}_{n=1}^{\infty} = \{1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \dots\} \Rightarrow 1 \leq \left\{n^{\frac{1}{n}}\right\} \leq \sqrt[3]{3},$$

Explanation:

$$3^{\frac{1}{3}} = 3^{\frac{4}{12}} = 3^{\frac{1}{3}} = (81)^{\frac{1}{12}} = 4^{\frac{1}{4}} = 4^{\frac{3}{12}} = 64^{\frac{1}{12}} \Rightarrow 81 > 64 \Rightarrow \sqrt[3]{3} > \sqrt[4]{4}$$

Thus,

i. the set of all upper bound $(\sqrt[3]{3}, \infty)$

ii. the smallest of all lower bound $(-\infty, 1]$

iii. the largest of all lower bound, $\text{glb} = 1$

iv. the smallest of all upper bound, $\text{lub} = \sqrt[3]{3}$

$$\left(3^{\frac{1}{3}}\right)^{\frac{1}{4}} \neq \left(4^{\frac{1}{4}}\right)^{\frac{1}{3}}$$

$$\frac{4}{3} \neq \frac{3}{4}$$

$$81 \neq 48$$

$$\{\ln(n)\}_{n=1}^{\infty} = \{0, \ln 2, \ln 3, \ln 4, \dots\}$$

i. no upper bound

ii. no lub

iii. the set of all lower bound $(-\infty, 0]$

iv. $\text{glb} = 0$

$$-1 \leq \{\cos n^2\}_{n=1}^{\infty} \leq 1$$

i. $[1, \infty)$

ii. $\text{lub} = 1$

iii. $(-\infty, -1)$

iv. $\text{glb} = -1$

$$0 \leq \{\sin^2(n)\}_{n=1}^{\infty} \leq 1, \text{ thus}$$

i. the set of all upper bound $[1, \infty)$

ii. $\text{lub} = 1$

iii. the set of all lower bound $(-\infty, 0]$

iv. $\text{glb} = 0$

$$m. \quad \left\{ \frac{3^n}{n!} \right\}_{n=1}^{\infty} = \left\{ 3, \frac{9}{2}, \frac{27}{6}, \frac{81}{24}, \dots, 0 \right\}_{n=1}^{\infty} \quad \text{from } n=3 \text{ to } (\infty) \text{ the}$$

sequence is decreasing $\frac{9}{2} \rightarrow 0 \therefore 0 < \left\{ \frac{3^n}{n!} \right\}_{n=1}^{\infty} \leq \frac{9}{2}$, thus

i. the set of all upper bound are $\left[\frac{9}{2}, \infty \right)$

ii. the smallest of all upper bound $\text{lub} = \frac{9}{2}$

iii. the set of all lower bound are $(-\infty, 0]$

iv. the largest of all lower bound (glb) is 0

n. **For odd number $n = 1, 3, 5, \dots, \infty$**

$$\bullet \left\{ 3^{\frac{1}{n}} (-1)^n \right\}_{n=1}^{\infty} = \left\{ -3, -3^{\frac{1}{3}}, -3^{\frac{1}{5}}, \dots, -1 \right\}$$

• **For even number, $n = 2, 4, 6, \dots$**

$$\left\{ 3^{\frac{1}{n}} (-1)^n \right\}_{n=2}^{\infty} = \left\{ 3^{\frac{1}{2}}, 3^{\frac{1}{4}}, 3^{\frac{1}{6}}, \dots, 1 \right\}$$

$$\therefore -3 \leq \left\{ 3^{\frac{1}{n}} (-1)^n \right\}_{n=1}^{\infty} < \sqrt{3} \text{ so that}$$

i. the set of all upper bounds are $[\sqrt{3}, \infty)$

ii. the **smallest** of all upper bounds (lub) $= \sqrt{3}$

iii. the set of all lower bound $(-\infty, -3]$

iv. The smallest of all upper bound (lub)

$$o. \quad 0 < \left\{ \frac{n!}{n^n} \right\} \leq 1 \leftarrow \text{exercise left for you}$$

Monotonic sequence

- Monotonicity is a property that is linked to the ordering of the real number system; simply stated a sequence is monotonic if it either preserves or reverses the order of the element its domain.

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Monotonic is either increasing or decreasing.

Definition: A sequence $\{a_n\}$ is called:

- Increasing if $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots$ i.e., $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.
- Strictly increasing if $a_1 < a_2 < a_3 < \dots < a_n < \dots$
- Decreasing if $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$
- Strictly decreasing if $a_1 > a_2 > a_3 > \dots > a_n > \dots$

Testing for monotonicity

Monotonic sequence $\{a_n\}$ can be identified by:

Difference between successive terms:

- $a_{n+1} - a_n \geq 0 \leftarrow$ increasing sequence
 - $a_{n+1} - a_n > 0 \leftarrow$ strictly increasing sequence
 - $a_{n+1} - a_n \leq 0 \leftarrow$ decreasing sequence
 - $a_{n+1} - a_n < 0 \leftarrow$ strictly decreasing sequence

Ratio test between successive terms

- $\frac{a_{n+1}}{a_n} \geq 1 \leftarrow$ increasing sequence
 - $\frac{a_{n+1}}{a_n} > 1 \leftarrow$ strictly increasing sequence
 - $\frac{a_{n+1}}{a_n} \leq 1 \leftarrow$ decreasing sequence
 - $\frac{a_{n+1}}{a_n} < 1 \leftarrow$ strictly decreasing

Note: It is better to use ratio test for exponential form of sequence. If a sequence $\{a_n\}$ is bounded and monotonic then it is convergent.

Illustrative Examples

2. Determine the given sequence $\{a_n\}$ is

i) monotonic

ii)

bounded

iii) convergent

a. $\left\{3 - \frac{1}{n}\right\}$

b. $\left\{\frac{n+2}{3n-2}\right\}$

c. $\left\{\frac{n}{2n+1}\right\}$

d. $\{3 + (-1)^n\}$

e. $\left\{4 + \frac{3}{n}\right\}$

f. $\{7\}_{n=1}^{\infty}$

Solution: Let $a_n = \left\{3 - \frac{1}{n}\right\}$ and $a_{n+1} = \left\{3 - \frac{1}{n+1}\right\}$

$$\Rightarrow a_{n+1} - a_n = 3 - \frac{1}{n+1} - \left(3 - \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{n+1} > 0$$

i) $\left\{3 - \frac{1}{n}\right\}$ is strictly increasing

ii) $2 \leq \left\{3 - \frac{1}{n}\right\} < 3$ bounded

iii) converge to 3 because increasing bounded sequence converges to lub

b. $\left\{\frac{n+2}{3n-2}\right\}_{n=1}^{\infty} = \left\{3, 1, \frac{5}{7}, \frac{3}{5}, \dots, \frac{1}{3}\right\}$ using difference b/n

successive terms. $1 - 3 < 0$, $\frac{5}{7} - 1 < 0 \dots$

$\Rightarrow$ for all n , $a_{n+1} - a_n < 0$. $\therefore$ i) $\left\{\frac{n+2}{3n-2}\right\}_{n=1}^{\infty}$ is strictly decreasing

ii) it is bounded between $\frac{1}{3}$ and 3

iv) since it is decreasing and bounded, so that it converges to $\text{glb } \frac{1}{3}$

c. $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \dots, \frac{1}{2}\right\} \Rightarrow \frac{2}{5} - \frac{1}{3} > 0, \frac{3}{7} - \frac{2}{5} > 0$
 $\therefore$ for all $n \geq 1$, $a_{n+1} - a_n > 0$

$\therefore$ As n increase from $1 \rightarrow \infty$, then $\left\{\frac{n}{2n+1}\right\}$ increase from $\frac{1}{3}$ to $\frac{1}{2}$
 thus

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i) $\left\{ \frac{n}{2n+1} \right\}_{n=1}^{\infty}$ strictly increasing

ii) $\frac{1}{3} \leq \left\{ \frac{n}{2n+1} \right\} < \frac{1}{2} \leftarrow$ it is bounded

iii) bounded increasing sequence are converges to lub $\frac{1}{2}$

d. $\{3 + (-1)^n\}_{n=1}^{\infty} = \{2, 4, 2, 4, \dots\} \leftarrow$ oscillating b/n 2 and 4

i) it is not monotonic (neither increasing nor decreasing)

ii) it is bounded $2 \leq \{3 + (-1)^n\}_{n=1}^{\infty} \leq 4$

iii) not convergent, because no monotonic

e. Let $a_n = \left\{ 4 + \frac{3}{n} \right\}$ and $a_{n+1} = \left\{ 4 + \frac{3}{n+1} \right\}$ then $\Rightarrow a_{n+1} < a_n$

$= 4 + \frac{3}{n+1} - \left(4 + \frac{3}{n} \right) = \frac{3}{n+1} - \frac{3}{n} < 0$ for all $n \geq 1$ so that

i) $\left\{ 4 + \frac{3}{n} \right\}_{n=1}^{\infty}$ is strictly decreasing

ii) $4 < \left\{ 4 + \frac{3}{n} \right\}_{n=1}^{\infty} \leq 7 \leftarrow$ it is bounded

iii) convergent, bounded and decreasing sequence are converge to glb 4

f. Let $a_n = \{7\}_{n=1}^{\infty} = \{7, 7, 7, \dots, 7\}$

i) constant sequence are monotonic (either increasing or decreasing)

ii) constant sequence are bounded

iii) constant sequence convergent

Determine the given sequence $\{a_n\}$ is

i) monotonic ii) bounded iii) convergent

a. $\left\{ \sin \frac{\pi}{3n} \right\}$ b. $\left\{ \sin \left(\frac{n\pi}{6} \right) \right\}$ c. $\left\{ \frac{\cos n}{n} \right\}$

Solution:

$$a. \left\{ \sin\left(\frac{\pi}{3n}\right) \right\} = \left\{ \sin\frac{\pi}{3}, \sin\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{9}\right) \dots 0 \right\}$$

i) it is decreasing sequence

$$ii) 0 \leq \left\{ \sin\left(\frac{\pi}{3n}\right) \right\} \leq \frac{\sqrt{3}}{2} \text{ is bounded}$$

iii) convergent

$$b. \left\{ \sin\left(\frac{n\pi}{6}\right) \right\} \leftarrow \text{it is oscillating b/n } -1 \text{ and } 1$$

$$i) \text{ not monotonic sequence } ii) -1 \leq \left\{ \sin\left(\frac{\pi}{6}\right) \right\} \leq 1 \text{ bounded b/n}$$

iii) not convergent

$$c. \left\{ \frac{\cos n}{n} \right\}_{n=1}^{\infty} \leftarrow \text{exercise left for you}$$

4. Determine the sequence $\{a_n\}$ is:

i) monotonic

ii) bounded

iii) convergent

$$a. \left\{ \left(\frac{2}{3}\right)^n \right\} \quad b. \left\{ \left(\frac{3}{2}\right)^n \right\} \quad c. \left\{ \frac{4^n}{n!} \right\} \quad d. \left\{ \frac{3^{n+1}}{4^{n-2}} \right\} \quad e. \{n e^{-n}\}$$

Solution: using ratio test

$$a. \text{ Let } a_n = \left(\frac{2}{3}\right)^n \text{ and } a_{n+1} = \left(\frac{2}{3}\right)^{n+1} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^n}$$

$$= \frac{\left(\frac{2}{3}\right)^n \cdot \frac{2}{3}}{\left(\frac{2}{3}\right)^n} = \frac{2}{3} < 1 \Rightarrow \frac{a_{n+1}}{a_n} < 1 \Rightarrow \left\{ \left(\frac{2}{3}\right)^n \right\} \text{ is strictly decreasing}$$

ii) it is bounded b/n $0 < \left\{ \left(\frac{2}{3} \right)^n \right\} \leq \frac{2}{3}$

iii) since it is bounded and decreasing so that it converges to 0

Let $a_n = \left(\frac{3}{2} \right)^n$, and $a_{n+1} = \left(\frac{3}{2} \right)^{n+1}$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{\left(\frac{3}{2} \right)^{n+1}}{\left(\frac{3}{2} \right)^n} = \frac{\left(\frac{3}{2} \right)^n \cdot \left(\frac{3}{2} \right)}{\left(\frac{3}{2} \right)^n} = \frac{3}{2} > 1$$

- i) it is strictly increasing ii) not bounded above
iii) not convergent

Let $a_n = \left\{ \frac{4^n}{n!} \right\}$ and $a_{n+1} = \left\{ \frac{4^{n+1}}{(n+1)!} \right\} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{4^n \cdot 4}{(n+1)(n!) \cdot 4^n}$

$\frac{4}{n+1} < 1$, for all $n \geq 3 \Rightarrow \frac{a_{n+1}}{a_n} < 1$, for all $n > 3$,

i) $\left\{ \frac{4^n}{n!} \right\}$ is eventually decreasing b/c the first three terms, a_1, a_2 and a_3 are increasing but $a_4, a_5, \dots, a_n$ is decreasing.

bounded b/n $0 < \left\{ \frac{4^n}{n!} \right\} \leq \frac{64}{6}$, since it is maximum when $n = 3$

convergent, converges to 0

$$\left\{ \frac{3^{n+1}}{4^{n+2}} \right\} = \left\{ \left(\frac{3}{4} \right)^n (48) \right\} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{3}{4} < 1 \Rightarrow \text{i) it is decreasing}$$

ii) bounded

iii) convergent to 0

Let $a_n = ne^{-2n}$, and $a_{n+1} = (n+1)e^{-2n-2}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)e^{-2n} \cdot e^{-2}}{(n)e^{-2n}} = e^{-2} \left(\frac{n+1}{n} \right) < 1 \Rightarrow \text{i) it is decreasing}$$

bounded b/n $0 < \{ne^{-2n}\} \leq e^{-2}$

iii) convergent to 0

Remark:

- Convergent sequence may not be monotonic.

Example: $\left\{ \frac{(-1)^n}{n} \right\}, \left\{ \frac{\sin(n)}{n} \right\}, \left\{ \frac{\cos n}{n} \right\}$, etc

- Bounded sequence may not be convergent. For example: $\{(-1)^n\}$, $\{\sin(n)\}$ etc.
- We can't find convergent sequence that is not bounded.

Limits of sequence

Limits of sequence describe:

- How the terms of sequence $\{a_n\}$ behaves as n goes to extremely very large number (n goes to infinity).
- What happen the sequence $\{a_n\}$ as n goes to infinity.

Definition: Limit of sequence is the behavior of sequence $\{a_n\}$ as n goes to infinity.

If L is the limit of sequence $\{a_n\}$ then we write symbolically,

$\lim_{n \rightarrow \infty} a_n = L \leftarrow$ Meaning the term of the sequence become unique number L as n goes to infinity.

Note: the limit value of sequence is unique if it exists.

If $\lim_{n \rightarrow \infty} a_n = 0$, then the sequence $\{a_n\}$ is called **null-sequence**.

Illustrative Examples

5. Decide whether or not each of the following sequence is null-sequence.

a. $\left\{ \frac{1}{n+2} \right\}$ b. $\left\{ 3 - \frac{3}{n+2} \right\}$ c. $\left\{ \frac{3n-2}{n^2+2} \right\}$ d. $\left\{ \frac{\cos n}{n} \right\}$

Solution: To decide evaluate $\lim_{n \rightarrow \infty} a_n$

a. $\lim_{n \rightarrow \infty} \left\{ \frac{1}{n+2} \right\} = 0 \leftarrow$ null sequence

b. $\lim_{n \rightarrow \infty} \left\{ 3 - \frac{3}{n+2} \right\} = 3 - \frac{3}{\infty} = 3 - 0 = 3 \leftarrow$ not null-sequence

$$\lim_{n \rightarrow \infty} \left\{ \frac{3n-2}{n^2+2} \right\} = \lim_{n \rightarrow \infty} \left\{ \frac{n \left(3 - \frac{2}{n} \right)}{n^2 \left(1 + \frac{2}{n^2} \right)} \right\} = \lim_{n \rightarrow \infty} \left\{ \frac{3 - \frac{2}{n}}{n \left(1 + \frac{2}{n^2} \right)} \right\} = 0$$

c. null sequence

$$\lim_{n \rightarrow \infty} \frac{\cos n}{n} = \frac{\text{finite}}{\text{infinite}} = 0 \leftarrow \text{null sequence}$$

Relation between monotonic, bounded convergence and the limit value of sequence

- A bounded monotonic sequence $\{a_n\}$ are convergent.
- If the sequence is increasing and bounded then the limit value of sequence is **the least upper bound** of sequence i.e., $\lim_{n \rightarrow \infty} a_n = \text{Lub}$
- If the sequence is decreasing and bounded then the limit value of the sequence $\{a_n\}$ is **the greatest lower bound** of sequence i.e., $\lim_{n \rightarrow \infty} a_n = \text{glb}$.

Note: To memorize easily study the following.

a. $\left\{ \frac{1}{n} \right\}, \left\{ \frac{1}{n+2} \right\}, \left\{ \frac{-3}{n+1} \right\}, \left\{ \frac{-1}{2(2n+3)} \right\} \leftarrow \text{increasing}$

b. $\left\{ \frac{1}{n} \right\}, \left\{ \frac{1}{n+2} \right\}, \left\{ \frac{4}{n+5} \right\}, \text{etc}$ all such forms are decreasing

c. $\left\{ 3 - \frac{1}{n} \right\}, \left\{ 2 - \frac{1}{n+2} \right\}, \dots$ etc all such form are increasing

d. $\left\{ 3 + \frac{1}{n} \right\}, \left\{ 2 + \frac{1}{n+2} \right\}, \left\{ 5 + \frac{3}{n+1} \right\}, \dots$ etc all such forms are decreasing.

Illustrative Example

6. Identify those sequences that are:
- monotonic
 - bounded
 - determine the limit and convergence point in terms of the glb and lub

A. $\left\{5 - \frac{3}{n+1}\right\}_{n=1}^{\infty}$ b. $\left\{2 + \frac{1}{n}\right\}_{n=1}^{\infty}$ c. $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$ d. $\left\{\frac{n}{4n-1}\right\}_{n=1}^{\infty}$
 e. $\{2(-1)^n\}_{n=1}^{\infty}$ f. $\left\{\frac{3-2n}{5n+6}\right\}_{n=1}^{\infty}$ g. $\left\{\frac{2-3n}{2n+1}\right\}_{n=1}^{\infty}$ h. $\left\{\frac{3n+2}{2n+1}\right\}_{n=1}^{\infty}$

Solution:

a. Let $a_n = 5 - \frac{3}{n+1}$ and $a_{n+1} = 5 - \frac{3}{n+2}$

$$\Rightarrow a_{n+1} - a_n = 5 - \frac{3}{n+2} - \left(5 - \frac{3}{n+1}\right) = \frac{3}{n+1} - \frac{3}{n+2} > 0 \text{ for all } n \geq 1$$

i) $\left\{5 - \frac{3}{n+1}\right\}_{n=1}^{\infty}$ is strictly increasing

ii) For $n \geq 1$ the sequence is bounded b/n $\frac{7}{2} \leq \left\{5 - \frac{3}{n+1}\right\} < 5$

iii) $\lim_{n \rightarrow \infty} \left(5 - \frac{3}{n+1}\right) = 5 \leftarrow \text{lub, converges to 5 b/c increasing and bounded sequence converges to lub.}$

B. Let

$$a_n = \left\{2 + \frac{1}{n}\right\} \text{ and } a_{n+1} = 2 + \frac{1}{n+1} \Rightarrow a_{n+1} - a_n = 2 + \frac{1}{n+1} - \left(2 + \frac{1}{n}\right) = \frac{1}{n+1} - \frac{1}{n} < 0 \text{ for all } n \geq 1, \text{ thus}$$

i) $\left\{2 + \frac{1}{n}\right\}$ is strictly decreasing

ii) For $n \geq 1$ it is bounded b/n $2 < \left\{2 + \frac{1}{n}\right\} \leq 3$

iii) $\lim_{n \rightarrow \infty} \left\{2 + \frac{1}{n}\right\} = 2 \leftarrow \text{glb, and also converges to 2. Since it is decreasing and bounded so that, the limit value, glb, and convergent point are the same.}$

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C. By long division $\left\{ \frac{n}{2n+1} \right\} = \left\{ \frac{1}{2} - \frac{1}{2(2n+1)} \right\}$ thus it is increasing.

i) it is bounded b/n $\frac{1}{3} \leq \left\{ \frac{n}{2n+1} \right\} < \frac{1}{2}$

ii) since it is increasing and bounded so that the limit value, lub, convergent point is the same $\frac{1}{2}$. i.e., $\lim_{n \rightarrow \infty} \left(\frac{n}{2n+1} \right) = \text{lub} = \text{convergent point} = \frac{1}{2}$.

D. By long division $\left\{ \frac{n}{4n-1} \right\} = \left\{ \frac{1}{4} + \frac{1}{4(4n-1)} \right\}$ is decreasing. Hence

$\left\{ \frac{n}{4n-1} \right\} = \left\{ \frac{1}{4} + \frac{1}{4(4n-1)} \right\}$ is decreasing.

i) bounded between $\frac{1}{4} < \left\{ \frac{n}{4n-1} \right\} \leq \frac{1}{3}$ for $n \geq 1$

ii) $\left\{ \frac{n}{4n-1} \right\}$ is bounded and decreasing and hence it converges.

Also, $\lim_{n \rightarrow \infty} \left(\frac{n}{4n-1} \right) = \text{glb} = \text{convergent point} = \frac{1}{4}$

E. $\{2(-1)^n\}_{n=1}^{\infty} = \{-2, 2, -2, \dots\}$ it is oscillating b/n -2 and 2 it does not tend to unique number. Therefore,

i) not monotonic

ii) bounded

iii) $\text{glb} = -2$, $\text{lub} = 2$, it does not converge, does not have limit value, so the sequence.

• $\{2(-1)^n\}_{n=1}^{\infty}$ is diverges • $\lim_{n \rightarrow \infty} \{2(-1)^n\}$ does not exist

F. By long division $\left\{ \frac{3-2n}{5n+6} \right\}_{n=1}^{\infty} = \left\{ -\frac{2}{5} + \frac{27}{5(5n+6)} \right\}$ since

$\left\{ \frac{27}{5(5n+6)} \right\}$ is decreasing, thus

i) $\left\{ \frac{3-2n}{5n+6} \right\} = \left\{ -\frac{2}{5} + \frac{27}{5(5n+6)} \right\}$ is decreasing

ii) For all $n \geq 1$, it is bounded b/n $-\frac{2}{5} < \left\{ \frac{3-2n}{5n+6} \right\} < \frac{1}{11}$

iii) Also, $\lim_{n \rightarrow \infty} \left\{ \frac{3-2n}{5n+6} \right\} = \text{glb} = \text{convergent point} = -\frac{2}{5}$ since it is decreasing and bounded.

G. $\left\{ \frac{2-3n}{2n+1} \right\} \leftarrow$ exercise left for you!

H. $\left\{ \frac{3n+2}{2n+1} \right\} \leftarrow$ exercise left for you!

Note: To memorize easily study the following.

a. $\left\{ 2^{\frac{1}{n}} \right\}, \left\{ 7^{\frac{1}{n}} \right\}, \left\{ e^{\frac{1}{n}} \right\}, \left\{ a^{\frac{1}{n}} \right\} \dots$ etc all those forms are decreasing, bounded and convergent also limit value = glb = convergent point = 1

b. $\frac{a^n}{n!}$ is strictly decreasing after $n = a - 1$ $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$ and

$0 < \frac{a^n}{n!} \leq \frac{a^{a-1}}{a!}$, note that a is positive integer.

Illustrative Examples

7. Determine the monotonicity, glb, lub and limit value of the following.

a. $\left\{ 4^{\frac{1}{n}} \right\}$ b. $\left\{ \left(\frac{3}{5} \right)^n \right\}$ c. $\left\{ \left(\frac{5}{3} \right)^n \right\}$ d. $\left\{ n^{\frac{1}{n}} \right\}$

e. $\left\{\frac{4^n}{n!}\right\}$ f. $\left\{\left(\frac{8^n}{n!}\right)\right\}$ g. $\left\{\frac{10^n}{n!}\right\}$
 h. $\left\{\left(\frac{5}{6}\right)^{n-2}\right\}$

Solution:

a. $\left\{4^n\right\}$ is decreasing with in $1 < \left\{4^n\right\} \leq 4$

• $\text{glb} = 1, \text{lub} = 4, \lim_{n \rightarrow \infty} 4^{\frac{1}{n}} = 4^0 = 1$

b. $\left\{\left(\frac{3}{5}\right)^n\right\}$ is decreasing with in $0 < \left\{\left(\frac{3}{5}\right)^n\right\} \leq \frac{3}{5}$ for all

$n \geq 1, \lim_{n \rightarrow \infty} \left\{\left(\frac{3}{5}\right)^n\right\} = \text{glb} = 0$, and $\text{lub} = \frac{3}{5}$ since the

denominator $5 > 3$, so that the rate of 5 is faster than 3, therefore the limit value tends to 0.

c. $\left\{\left(\frac{5}{3}\right)^n\right\}$ is increasing with $\frac{5}{3} \leq \left(\frac{5}{3}\right)^n < \infty$ for all $n \geq 1$,

thus it is not bounded above, but bounded below.

$\therefore \text{glb} = \frac{5}{3} \therefore \lim_{n \rightarrow \infty} \left(\frac{5}{3}\right)^n$ does not exist $\Rightarrow$ it diverges.

d. $\left\{n^{\frac{1}{n}}\right\}$ is decreasing within $1 \leq \left\{n^{\frac{1}{n}}\right\} \leq \sqrt[3]{3}$ for all $n \geq 1$,

$\therefore \text{lub} = \sqrt[3]{3}, \text{glb} = \lim_{n \rightarrow \infty} \left(n^{\frac{1}{n}}\right) = 1$

e. $\{ne^{-2n}\}_{n=1}^{\infty} = \left\{ \frac{n}{e^2}, \frac{n}{e^4}, \frac{n}{e^6}, \dots, \frac{n}{e^{2n}} \right\} \leftarrow \text{decreasing}$

$$0 < \{ne^{-2n}\} \leq \frac{1}{e^2} \therefore \text{lub} = \frac{1}{e^2}, \lim_{n \rightarrow \infty} \{ne^{-2n}\} = \text{glb} = 0$$

f. $\left\{ \frac{4^n}{n!} \right\}$ even if the first three terms are increasing, $n \geq 4$, the sequence is eventually decreasing.

$$0 < \left\{ \frac{4^n}{n!} \right\} \leq \frac{4^3}{3!} \Rightarrow 0 < \left\{ \frac{4^n}{n!} \right\} \leq \frac{32}{3} \therefore \text{lub} = \frac{32}{3}, \lim_{n \rightarrow \infty} \frac{4^n}{n!} = 0$$

To show this use ratio

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}}{(n+1)(n!)} \cdot \frac{n!}{4^n} = \frac{4}{n+1} < 1, \text{ for all } n \geq 4,$$

show the sequence $\left\{ \frac{4^n}{n!} \right\}$ is decreasing for all $n \geq 4$.

g. $\left\{ \frac{8^n}{n!} \right\}$ even though the first seven terms are increasing,

but for all $n \geq 8$ the sequence $\left\{ \frac{8^n}{n!} \right\}$ eventually decreases

$$\text{with in } 0 < \left\{ \frac{8^n}{n!} \right\} \leq \frac{8^7}{7!} \therefore \text{lub} = \frac{8^7}{7!}, \lim_{n \rightarrow \infty} \frac{8^n}{n!} = 0$$

h. $\left\{ \frac{10^n}{n!} \right\} \leftarrow \text{exercise left for you.}$

i. $\{3^{1-n}\} = \{3^{-n}(3)^1\} = \left\{ \frac{3}{3^n} \right\}$ it is decreasing

$$0 < \left\{ \frac{3}{3^n} \right\} \leq 1,$$

$$\text{for all } n \geq 1. \therefore \text{lub} = 1, \lim_{n \rightarrow \infty} 3^{1-n} = 0 \leftarrow \text{glb}$$

i. $\left\{\left(\frac{5}{6}\right)^{n-2}\right\}$ ← exercise left for you.

Definition: The sequence $\{a_n\}$ is said to **converge** if there is a number L such that $\lim_{n \rightarrow \infty} a_n = L$. If there is no such number, then the sequence is **diverge**.

Illustrative Examples

Determine wheatear the sequence converge or diverge.

a. $\left\{\frac{n}{n+2}\right\}_{n=1}^{\infty}$

b. $\left\{\frac{3n+1}{2-4n}\right\}_{n=1}^{\infty}$

c. $\left\{(-1)^n \left(\frac{n}{2n+1}\right)\right\}_{n=1}^{\infty}$

d. $\left\{(-1)^n \left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$

e. $\left\{2 + (-1)^n\right\}_{n=1}^{\infty}$

f. $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

g. $\{8-2n\}_{n=1}^{\infty}$

h. $\left\{\frac{(2n+1)^2}{3n^2+2n+1}\right\}_{n=1}^{\infty}$

Solution: To simplify finding the limit by divide the numerator and denominator by n .

a. $\lim_{n \rightarrow \infty} \frac{n}{n+2} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n}} = \frac{1}{1+0} = 1 \leftarrow \text{converges}$

b. $\lim_{n \rightarrow \infty} \frac{3n+1}{2-4n} = \lim_{n \rightarrow \infty} \frac{3 + \frac{1}{n}}{\frac{2}{n} - 4} = \frac{3+0}{0-4} = \frac{-3}{4} \leftarrow \text{converges}$

c. $\lim_{n \rightarrow \infty} (-1)^n \left(\frac{n}{2n+1}\right) = \lim_{n \rightarrow \infty} (-1)^n \left(\frac{1}{2 + \frac{1}{n}}\right) = -\frac{1}{2} \text{ or } \frac{1}{2} \leftarrow \text{diverge}$

b/c unique. $\therefore \lim_{n \rightarrow \infty} (-1)^n \left(\frac{n}{2n+1}\right)$ doesn't exist so that diverge.

d. $\lim_{n \rightarrow \infty} (-1)^n \frac{1}{n} = 0 \leftarrow$ converge e. $\lim_{n \rightarrow \infty} \{2 + (-1)^n\}$

• For n is odd term, $\lim_{n \rightarrow \infty} \{2 + (-1)^n\} = 1$

• For n is even term $\lim_{n \rightarrow \infty} \{2 + 1\} = 3$, since the limit value is not unique

$\therefore \lim_{n \rightarrow \infty} \{2 + (-1)^n\}$ doesn't exist, so that diverge.

f. $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0 \leftarrow$ diverge

g. $\lim_{n \rightarrow \infty} \{8 - 2n\} = -\infty$, since it is not finite number, therefore diverges.

h. $\lim_{n \rightarrow \infty} \frac{(2n+1)^2}{3n^2 + 2n + 1} = \lim_{n \rightarrow \infty} \frac{4n^2 + 4n + 1}{3n^2 + 2n + 1}$, dividing numerator and denominator by n^2

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{4 + \frac{1}{n} + \frac{1}{n^2}}{3 + \frac{2}{n} + \frac{1}{n^2}} = \frac{4 + 0 + 0}{3 + 0 + 0} = \frac{4}{3} \leftarrow \text{converge}$$

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be convergent sequence with $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = k$ then, $a_n + b_n$, $a_n - b_n$, ca_n , $(a_n b_n)$ and $\frac{a_n}{b_n}$, for $b_n \neq 0$, for every n are convergent with:

i. $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = L + k$

ii. $\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n = L - k$

iii. $\lim_{n \rightarrow \infty} (ca_n) = c \lim_{n \rightarrow \infty} a_n = cL$

iv. $\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n = Lk$

v. $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} = \frac{L}{k}$

vi. $\lim_{n \rightarrow \infty} (a_n)^m = \left(\lim_{n \rightarrow \infty} a_n \right)^m = L^m$

Illustrative Examples

Evaluate each of the following limit:

a. $\lim_{n \rightarrow \infty} \frac{3n+4}{5-6n}$

c. $\lim_{n \rightarrow \infty} \left(\frac{2n+1}{3n+4} \right) \left(\frac{4n-1}{7n-1} \right)$

b. $\lim_{n \rightarrow \infty} \frac{6n^2+5n+7}{4n^2-8n+1}$

d. $\lim_{n \rightarrow \infty} [\ln(3n+2) - \ln(n-5)]$

e. $\lim_{n \rightarrow \infty} \frac{1-5n^4}{2n^4+8n^3}$

f. $\lim_{n \rightarrow \infty} \left(\frac{2n+(-1)^n}{n} \right)$

Solution: Divide the numerator and denominator by larger exponent of n .

a. $\lim_{n \rightarrow \infty} \frac{3n+4}{5-6n} = \lim_{n \rightarrow \infty} \frac{3 + \frac{4}{n}}{\frac{5}{n} - 6} = \frac{3+0}{0-6} = \frac{-3}{6} = -\frac{1}{2}$

$$b. \lim_{n \rightarrow \infty} \frac{6n^2 + 5n + 7}{4n^2 - 8n + 1} = \lim_{n \rightarrow \infty} \frac{\left(6 + \frac{5}{n} + \frac{7}{n^2}\right)}{4 - \frac{8}{n} + \frac{7}{n^2}} = \frac{6+0+0}{4-0+0} = \frac{6}{4} = \frac{3}{2}$$

$$c. \lim_{n \rightarrow \infty} \left(\frac{2n+1}{3n+4} \right) \cdot \left(\frac{4n-1}{7n+1} \right) = \lim_{n \rightarrow \infty} \frac{2n+1}{3n+4} \cdot \lim_{n \rightarrow \infty} \frac{4n-1}{7n+1}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{2 + \frac{1}{n}}{3 + \frac{4}{n}} \right) \lim_{n \rightarrow \infty} \left(\frac{4 + \frac{1}{n}}{7 + \frac{1}{n}} \right) = \left(\frac{2+0}{3+0} \right) \left(\frac{4+0}{7+0} \right) = \frac{2}{3} \cdot \frac{4}{7} = \frac{8}{21}$$

$$d. \lim_{n \rightarrow \infty} [\ln(3n+2) - \ln(5n+6)]$$

$$= \lim_{n \rightarrow \infty} \ln \left(\frac{3n+2}{5n+6} \right) = \ln \left(\frac{3}{5} \right) = \ln 3 - \ln 5$$

$$e. \lim_{n \rightarrow \infty} \frac{1-5n^4}{2n^4+8n^3} = \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n^4} - 5}{2 + \frac{8}{n}} \right) = \frac{0-5}{2+0} = \frac{-5}{2} \text{ [divide by } n^4]$$

$$f. \lim_{n \rightarrow \infty} \left(\frac{2n + (-1)^n}{n} \right) = \lim_{n \rightarrow \infty} \left(2 + \frac{(-1)^n}{n} \right) = 2 + 0 = 2$$

Limit involving arithmetic and geometric sequence

The limit of arithmetic and geometric sequence arise frequently. The method of finding is first determine the n th partial sum of the sequence, then evaluate the limit value.

$a_n = S_n = \frac{n}{2} [A_1 + A_n] \leftarrow$ n th partial sum of arithmetic sequence

$a_n = G_1 \left[\frac{r^n - 1}{r - 1} \right] \leftarrow$ n th partial sum of G.P.

Illustrative Examples

Find the limit of each sequence:

$$a. \lim_{n \rightarrow \infty} \left\{ \frac{2+5+8+\dots+3n-1}{1-n^2} \right\} \quad b. \lim_{n \rightarrow \infty} \left\{ \frac{1+2+3+\dots+n}{3-4n^2} \right\}$$

$$c. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4i-3}{5-4n^2}$$

$$d. \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{3k+2}{2n^2-1}$$

$$e. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2i-1}{4-6n^2}$$

$$f. \lim_{n \rightarrow \infty} \left[\frac{1+2^2+3^2+\dots+n^2}{n^3} \right]$$

$$g. \lim_{n \rightarrow \infty} \left\{ 5 + \frac{5}{4} + \frac{5}{4^2} + \dots + \frac{5}{4^{n-1}} + \dots \right\}$$

$$h. \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n \left(-\frac{1}{2} \right)^i \right\}$$

$$i. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4}{2^i-1}$$

$$j. \lim_{n \rightarrow \infty} \left\{ \sum_{k=0}^n \left(\frac{5}{2^k} - \frac{1}{3^k} \right) \right\}$$

Solution: First determine the n^{th} partial sum.

$$a. \lim_{n \rightarrow \infty} \left\{ \frac{2+5+8+\dots+3n-1}{1-n^2} \right\} = \lim_{n \rightarrow \infty} \left[\frac{\frac{n}{2}(2+3n-1)}{1-n^2} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{3n^2+n}{2-2n^2} \Rightarrow \lim_{n \rightarrow \infty} \frac{3+\frac{1}{n}}{\frac{2}{n^2}-2} = \frac{3+0}{0-2} = -\frac{3}{2} \leftarrow \text{Dividing by } n^2$$

$$b. \lim_{n \rightarrow \infty} \left\{ \frac{1+2+3+\dots+n}{3-4n^2} \right\} = \lim_{n \rightarrow \infty} \left\{ \frac{n}{2} \left(\frac{1+n}{3-4n^2} \right) \right\} = \lim_{n \rightarrow \infty} \frac{n^2+n}{6-8n^2}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{\frac{6}{n^2}-8} = \frac{1+0}{0-8} = -\frac{1}{8} \leftarrow \text{dividing by } n^2$$

$$c. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4i-3}{5-4n^2} = \lim_{n \rightarrow \infty} \frac{n}{2} \left(\frac{1+4n-3}{5-4n^2} \right), \text{ use } A_1 = 1, A_n = 4n-3$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{2n^2 - 2n}{5 - 4n^2} \right) = \lim_{n \rightarrow \infty} \frac{\left(2 - \frac{2}{n} \right)}{\frac{n^2}{5} - 4} = \frac{2-0}{0-4} = \frac{-2}{-4} = \frac{1}{2}$$

$$d. \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{3k+2}{2n^2-1} = \lim_{n \rightarrow \infty} \frac{n}{2} \left[\frac{5+3n+2}{2n^2-1} \right], \text{ use } A_1 = 5, A_n = 3n+2$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left[\frac{3n^2 + 7n}{4n^2 - 2} \right] = \lim_{n \rightarrow \infty} \frac{3 + \frac{7}{n}}{4 - \frac{2}{n^2}} = \frac{3+0}{4-0} = \frac{3}{4}$$

$$e. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2i-1}{4-6n^2} \leftarrow \text{exercise left for you}$$

$$f. \text{ Remember that: } 1^2 + 2^2 + 3^2 + \dots + n^2 + \dots$$

$$= \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{1}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \dots + \frac{n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{2}{6} = \frac{1}{3}$$

$$g. \lim_{n \rightarrow \infty} \left\{ 5 + \frac{5}{4} + \frac{5}{4^2} + \dots + \frac{5}{4^{n-1}} + \dots \right\} \text{ we have, } G_1 = 5, r = \frac{1}{4}$$

$$\therefore a_n = S_n = \frac{G_1[1-r^n]}{1-r} = \frac{5 \left[1 - \left(\frac{1}{4} \right)^n \right]}{1 - \frac{1}{4}} = \frac{20 \left(1 - \left(\frac{1}{4} \right)^n \right)}{3}$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{20}{3} \left(1 - \left(\frac{1}{4} \right)^n \right) = \frac{20}{3} - 0 = \frac{20}{3}$$

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$$\sum_{i=1}^n \left(-\frac{1}{2}\right)^i = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \dots, G_1 = -\frac{1}{2}, r = \frac{-1}{2}$$

$$\therefore a_n = S_n = \frac{G_1[1-r^n]}{1-r} = \frac{-\frac{1}{2}\left[1-\left(\frac{-1}{2}\right)^n\right]}{1+\frac{1}{2}} = -\frac{1}{3}\left[1-\left(\frac{-1}{2}\right)^n\right]$$

$$\therefore \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(-\frac{1}{2}\right)^i = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left\{-\frac{1}{3}\right\} \left[1-\left(\frac{-1}{2}\right)^n\right] = -\frac{1}{3}$$

$$\sum_{i=1}^n \frac{4}{2^{i-1}} = 4 + \frac{4}{2} + \frac{4}{2^2} + \frac{4}{2^3} + \dots + \frac{4}{2^{n-1}}, G_1 = 4, r = \frac{1}{2}$$

$$\therefore a_n = S_n = \frac{G_1[1-r^n]}{1-r} = \frac{4\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = 8\left[1-\left(\frac{1}{2}\right)^n\right]$$

$$\therefore \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4}{2^{i-1}} = \lim_{n \rightarrow \infty} 8\left[1-\left(\frac{1}{2}\right)^n\right] = 8$$

$$\sum_{k=0}^n \left(\frac{5}{2^k} - \frac{1}{3^k}\right) = \sum_{k=0}^n \frac{5}{2^k} - \sum_{k=0}^n \frac{1}{3^k}$$

$$\frac{5\left[1-\left(\frac{1}{2}\right)^{n+1}\right]}{1-\frac{1}{2}} - \frac{1\left[1-\left(\frac{1}{3}\right)^{n+1}\right]}{1-\frac{1}{3}} = 10\left[1-\left(\frac{1}{2}\right)^{n+1}\right] - \frac{3}{2}\left[1-\left(\frac{1}{3}\right)^{n+1}\right]$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \left(\frac{5}{2^k} - \frac{1}{3^k}\right) = \lim_{n \rightarrow \infty} \left\{10\left[1-\left(\frac{1}{2}\right)^{n+1}\right]\right\} - \lim_{n \rightarrow \infty} \frac{3}{2}\left[1-\left(\frac{1}{3}\right)^{n+1}\right]$$

$$10 - \frac{3}{2} = \frac{17}{2}$$

Limit involving non-stopping decimal sequence**Illustrative Examples**

11. Find the limit of the following decimal.

a. $\{0.5, 0.55, 0.555, \dots\}$

b. $\{2.4, 2.44, 2.444, \dots\}$

c. $\{-3.6, -3.66, -3.666, \dots\}$

Solution:

a. $\{0.5, 0.55, 0.555, \dots\} = \{0.1, 0.11, 0.111, \dots\}$

$$= \frac{5}{9} \{0.9, 0.99, 0.999, \dots\}$$

$$= \frac{5}{9} \left\{ 1 - \frac{1}{10}, 1 - \frac{1}{10^2}, 1 - \frac{1}{10^3}, \dots, 1 - \frac{1}{10^n} \right\}$$

$$= \frac{5}{9} \left\{ 1 - \frac{1}{10^n} \right\} \therefore \lim_{n \rightarrow \infty} \frac{5}{9} \left\{ 1 - \frac{1}{10^n} \right\} = \frac{5}{9}$$

b. $\{2.4, 2.44, 2.444, \dots\} = 2 + 4 \{0.1, 0.11, 0.111, \dots\}$

$$\Rightarrow 2 + \frac{4}{9} \{0.9, 0.99, 0.999, \dots\} = 2 + \frac{4}{9} \left\{ 1 - \frac{1}{10}, 1 - \frac{1}{10^2}, \dots, 1 - \frac{1}{10^n} \right\}$$

$$\therefore \text{The } n\text{th term, } 2 + \frac{4}{9} \left\{ 1 - \frac{1}{10^n} \right\} \Rightarrow \lim_{n \rightarrow \infty} 2 + \frac{4}{9} \left\{ 1 - \frac{1}{10^n} \right\} = 2 + \frac{4}{9}$$

c. $\{-3.6, -3.66, -3.666, \dots\} = -3 + 6 \{0.1, 0.11, 0.111, \dots\}$

$$= -3 + \frac{-6}{9} \{0.9, 0.99, 0.999, \dots\}$$

$$= -3 + \frac{-6}{9} \left\{ 1 - \frac{1}{10}, 1 - \frac{1}{10^2}, \dots, 1 - \frac{1}{10^n} \right\}$$

$$\therefore \text{The } n\text{th term, } -3 + \frac{-6}{9} \left\{ 1 - \frac{1}{10^n} \right\} \Rightarrow \lim_{n \rightarrow \infty} \left(-3 + \frac{-6}{9} \left\{ 1 - \frac{1}{10^n} \right\} \right)$$

$$= -3 + \frac{-6}{9} = \frac{-11}{3}$$

Limit involving square root of sequence

12. Find the limit of the following sequence.

a. $\lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + 5}}{2n}$

b. $\lim_{n \rightarrow \infty} \frac{\sqrt{36n^2 + 1}}{11n}$

$$c. \lim_{n \rightarrow \infty} \sqrt[3]{\frac{n^3 + 4}{2n^3 + 5n}}$$

$$d. \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + 2} - 4}{3n + 2}$$

$$e. \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{\sqrt{n-1}}$$

$$f. \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 - 4n + 2}}{\sqrt{n^2 + 4n} - 4}$$

$$g. \lim_{n \rightarrow \infty} \frac{\sqrt{4n^2 + 1} + 4n}{\sqrt[3]{8n^3 + 6} + n}$$

$$h. \lim_{n \rightarrow \infty} \frac{\sqrt{4n^2 + 1} - \sqrt{n^2 + 1}}{n}$$

$$i. \lim_{n \rightarrow \infty} \frac{n^{1/3}(n^2 + 1)^{1/3}}{\sqrt{2n^2 + 3n + 1}}$$

$$j. \lim_{n \rightarrow \infty} \frac{\sqrt{1 + n^4} - (6 - n^2)}{n^2}$$

$$k. \lim_{n \rightarrow \infty} \sqrt{n^6 + 5n^3} - n^3$$

$$l. \lim_{n \rightarrow \infty} \sqrt{n^2 - 3n} - n$$

Solution: Here, we treat $\sqrt{n^2} - n$ and cancelling.

$$a. \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + 5}}{2n} = \frac{\lim_{n \rightarrow \infty} n^2 \left(1 + \frac{5}{n^2}\right)}{2n} = \frac{n \lim_{n \rightarrow \infty} \left(1 + \frac{5}{n^2}\right)}{2n} = \frac{1}{2}$$

$$b. \lim_{n \rightarrow \infty} \frac{\sqrt{3n^4 + n}}{n^2 - 8} = \frac{\lim_{n \rightarrow \infty} n^4 \left(3 + \frac{1}{n^3}\right)}{n^2 \left(1 - \frac{8}{n^2}\right)}$$

$$= \frac{n^2 \lim_{n \rightarrow \infty} \left(3 + \frac{1}{n^3}\right)}{n^2 \left(1 - \frac{8}{n^2}\right)} = \frac{\sqrt{3}}{1 - 0} = \sqrt{3}$$

$$c. \lim_{n \rightarrow \infty} \sqrt[3]{\frac{n^3 + 4}{2n^3 + 5n}} = \sqrt[3]{\frac{\lim_{n \rightarrow \infty} n^3 \left(1 + \frac{4}{n^3}\right)}{\lim_{n \rightarrow \infty} n^3 \left(2 + \frac{5}{n^2}\right)}} = \sqrt[3]{\frac{1 + 0}{2 + 0}} = \sqrt[3]{\frac{1}{2}}$$

$$d. \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + 2} - 4}{3n + 2} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 \left(1 + \frac{2}{n^2}\right)} - 4}{n \left(3 + \frac{2}{n}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{n \left(\sqrt{1 + \frac{2}{n^2}} - \frac{4}{n} \right)}{n \left(3 + \frac{2}{n} \right)} = \frac{1 + 0 - 0}{3 + 0} = \frac{1}{3}$$

$$e. \lim_{n \rightarrow \infty} \frac{\sqrt{n} + 1}{\sqrt{n} - 1} = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{\sqrt{n}}\right)}{1 - \frac{1}{\sqrt{n}}} = \frac{1 + 0}{1 - 0} = 1$$

$$f. \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 - 4} + 2}{\sqrt{n^2 + 4n} - 4} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 \left(1 - \frac{4}{n^2}\right)} + 2}{\sqrt{n^2 \left(1 + \frac{4}{n}\right)} - 4}$$

$$= \lim_{n \rightarrow \infty} \frac{n \left(\sqrt{1 - \frac{4}{n^2}} + \frac{2}{n} \right)}{n \left(1 + \frac{4}{n} - \frac{4}{n} \right)} = \frac{\sqrt{1 - 0} + 0}{\sqrt{1 + 0 - 0}} = 1$$

$$g. \lim_{n \rightarrow \infty} \frac{\sqrt{4n^2 + 1} + 4n}{\sqrt[3]{8n^3 + 6} + n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 \left(4 + \frac{1}{n^2}\right)} + 4n}{\sqrt[3]{n^3 \left(8 + \frac{6}{n^3}\right)} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{n \left(\sqrt{4 + \frac{1}{n^2}} + 4 \right)}{n \left(\sqrt[3]{8 + \frac{6}{n^3}} + 1 \right)} = \frac{\sqrt{4 + 0} + 4}{2 + 1} = \frac{6}{3} = 2$$

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$$\begin{aligned}
 \text{h. } \lim_{n \rightarrow \infty} \frac{\sqrt{4n^2 + 1} - \sqrt{n^2 + 1}}{n} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 \left(4 + \frac{1}{n^2}\right)} - \sqrt{n^2 \left(1 + \frac{1}{n^2}\right)}}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{n \left(\sqrt{4 + \frac{1}{n^2}} - \sqrt{1 + \frac{1}{n^2}} \right)}{n} = \sqrt{4 + 0} - \sqrt{1 + 0} = 2 - 1 = 1
 \end{aligned}$$

$$\begin{aligned}
 \text{i. } \lim_{n \rightarrow \infty} \frac{n^{1/3} (n^2 + 1)^{1/3}}{\sqrt{2n^2 + 3n + 1}} &= \lim_{n \rightarrow \infty} \frac{\sqrt[3]{n^3} \sqrt[3]{n^2 + 1}}{\sqrt{2n^2 + 3n + 1}} = \lim_{n \rightarrow \infty} \frac{\sqrt[3]{n^3} \sqrt[3]{1 + \frac{1}{n^2}}}{\sqrt{2 + \frac{3}{n} + \frac{1}{n^2}}} \\
 &\Rightarrow \lim_{n \rightarrow \infty} \frac{\sqrt[3]{n^3 \left(1 + \frac{1}{n^2}\right)}}{\sqrt{n^2 \left(2 + \frac{3}{n} + \frac{1}{n^2}\right)}} = \lim_{n \rightarrow \infty} \frac{n \sqrt[3]{1 + \frac{1}{n^2}}}{n \sqrt{2 + \frac{3}{n} + \frac{1}{n^2}}} = \frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{j. } \lim_{n \rightarrow \infty} \frac{\sqrt{1 + n^4} - (6 - n^2)}{n^2} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n^4 \left(1 + \frac{1}{n^4}\right)} - (6 - n^2)}{n^2} \\
 &= \lim_{n \rightarrow \infty} \frac{n^2 \left(\sqrt{1 + \frac{1}{n^4}} - \frac{6}{n^2} + 1 \right)}{n^2} = \sqrt{1 + 0} - 0 + 1 = 1 + 1 = 2
 \end{aligned}$$

k.

We treat each sequence a fraction denominator and rationalize the numerator

$$\begin{aligned}
 \Rightarrow \lim_{n \rightarrow \infty} \sqrt{n^6 + 5n^3} - n^3 &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^6 + 5n^3} - n^3)(\sqrt{n^6 + 5n^3} + n^3)}{\sqrt{n^6 + 5n^3} + n^3} \\
 &= \lim_{n \rightarrow \infty} \frac{n^6 + 5n^3 - n^6}{\sqrt{n^6 + 5n^3} + n^3}
 \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \frac{5n^3}{\sqrt{n^6 + 5n^3 + n^3}} = \lim_{n \rightarrow \infty} \frac{5n^3}{n^3 \left(\sqrt{1 + \frac{5}{n^3} + 1} \right)} \Rightarrow \frac{5}{\sqrt{1+0}-1} = \frac{5}{2}$$

1. $\lim_{n \rightarrow \infty} \sqrt{n^2 - 3n} - n \leftarrow$ exercise left for you. **Answer:** $-\frac{3}{2}$

Limit involving telescoping series

- The series of the form: $(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots$ is called **telescoping series**.

Note: a_2 is canceled by 2nd term

a_3 is canceled by 3rd term, and so on

- The n th partial sum of telescoping series is $S_n = a_1 - a_{n+1}$.

Illustrative Examples

13. Find the limit of the following sequence.

a. $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{1}{k} - \frac{1}{k+1} \right]$

b. $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{1}{4k^2 - 1} \right]$

c. $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{2}{k(k+2)} \right]$

d. $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{1}{(k+2)(k+3)} \right]$

Solution:

a. $\sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots$ is given

by $S_n = a_1 - a_{n+1} = 1 - \frac{1}{n+1}$.

$\therefore \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1$

A. $\sum_{k=1}^n \left[\frac{1}{4k^2 - 1} \right] = \sum_{k=1}^n \left[\frac{1}{(2k-1)(2k+1)} \right]$, using partial decomposition.

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$$\left[\frac{1}{(2k-1)(2k+1)} \right] = \frac{A}{2k-1} + \frac{B}{2k+1} \Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}$$

$$\Rightarrow \sum_{k=1}^n \left[\frac{1}{(2k-1)(2k+1)} \right] = \sum_{k=1}^n \left[\frac{1}{2(2k-1)} + \frac{-1}{2(2k+1)} \right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{1}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \right] = \frac{1}{2} \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \dots \right] \text{ give}$$

$$n \text{ by } S_n = a_1 - a_{n+1} = \frac{1}{2} \left(1 - \frac{1}{2n+1} \right).$$

$$\therefore \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{1}{4k^2 - 1} \right] = \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 - \frac{1}{2n+1} \right) = \frac{1}{2}$$

B. $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{2}{k(k+2)} \right] \leftarrow \text{exercise left for you Answer: 1}$

C. $\frac{1}{(k+2)(k+3)} = \frac{A}{k+2} + \frac{B}{k+3} \Rightarrow A = 1, B = -1$

$$\Rightarrow \sum_{k=1}^n \frac{1}{(k+2)(k+3)} = \sum_{k=1}^n \left(\frac{1}{k+2} - \frac{1}{k+3} \right)$$

$$= \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \dots \text{ is given by } S_n = a_1 - a_{n+1}$$

$$= \frac{1}{3} - \frac{1}{n+3}$$

$$\therefore \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[\frac{1}{(k+2)(k+3)} \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{3} - \frac{1}{n+3} \right] = \frac{1}{3}$$

Limit of Function

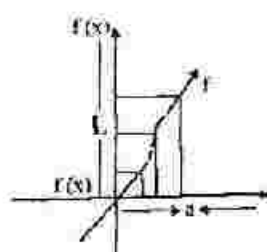
The limit of function describes how the function behaves as the independent variable approaches (goes to) certain a given value.

Limit of function at a point

The limit of function at a point:

- Describes the behavior of function near a particular point from either side of a point but may not be at the point itself.
- Investigate the value of function near particular point.

Definition: The intuitive definition of the limit of a function at a point.



Let $y = f(x)$ be function, and $a \in \mathbb{R}$. If $f(x)$ gets closer and closer to single real number ' L ' as x gets closer and closer to ' a ' (but may not equal to a), then we say that the limit of $f(x)$ as x close to a is L symbolically, this written as:

$$\lim_{x \rightarrow a} f(x) = L$$

One sided limit

$\lim_{x \rightarrow a} f(x) = L$ is called two sided limit because it requires the value of $f(x)$ get closer and closer to L as x get closer and closer to ' a ' from either of the sides. However, some function behave differently on the two sides when x close to ' a ', in such case we investigate limiting behavior whether the value of x near ' a ' from the left side or on the right side of ' a '.

Notation of the following terminology

Notation	Interpretation	Classification
$x \rightarrow a^+$	x close to 'a' from the right (slightly greater than 'a') $\Rightarrow \lim_{x \rightarrow a^+} f(x) = L$	right side limit or (one side limit)
$x \rightarrow a^-$	x close to 'a' from the left (slightly less than a) $\Rightarrow \lim_{x \rightarrow a^-} f(x) = L$	left side limit (one side limit)
$x \rightarrow a$	x close to 'a' from both side $\Rightarrow \lim_{x \rightarrow a} f(x) = L$	two side limit

Existence of two side limit $\left(\lim_{x \rightarrow a} f(x) = L \right)$

The two sided limit of a function $f(x)$ exist at ' a ' if and only if both of the one sided limit exist at ' a ' and have the same value, i.e.,

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$$

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Illustrative Examples

14. Find the limit:

- i) Which side of the limit exist?
 ii) Does the two side limit exist?

A. $\lim_{x \rightarrow 0} [x]$ b. $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$ c. $\lim_{x \rightarrow 2} [x]$

d. $\lim_{x \rightarrow 3} \sqrt{3-x}$ e. $\lim_{x \rightarrow 2} \sqrt{x^2-4}$ f. $\lim_{x \rightarrow 2} \sqrt{4-x^2}$

g. $\lim_{x \rightarrow -1} \sqrt{\sqrt{x^2-1}-x}$ h. $\lim_{x \rightarrow \frac{1}{2}} [x]$

Solution: Here $[]$ greatest integer function.

a. $\lim_{x \rightarrow 0^+} [x] = 0 \leftarrow$ limit from the right exist (x slightly greater than 0)

$\lim_{x \rightarrow 0^-} [x] = -1 \leftarrow$ limit from the left exist (x slightly less than 0)

two sided $\lim_{x \rightarrow 0} [x]$ does not exist b/c $0 \neq -1$

b. $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2} \Rightarrow \lim_{x \rightarrow 2^+} \frac{x-2}{x-2} = 1$ and $\lim_{x \rightarrow 2^-} \frac{-(x-2)}{x-2} = -1$

$\therefore \lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$ does not exist, since the one side (left and right) are not equal.

c. $\lim_{x \rightarrow 2} [x] \Rightarrow \lim_{x \rightarrow 2^+} [x] = 2$ and $\lim_{x \rightarrow 2^-} [x] = 1$

• Since the one side limit are not equal.

• The two side limit, $\lim_{x \rightarrow 2} [x]$ does not exist.

d. $\lim_{x \rightarrow 3} \sqrt{3-x} = 0$ but $\lim_{x \rightarrow 3^+} \sqrt{3-x}$ does not exist

e. $\lim_{x \rightarrow 2} \sqrt{x^2-4} \Rightarrow \lim_{x \rightarrow 2^+} \sqrt{x^2-4} = 0$ and $\lim_{x \rightarrow 2^-} \sqrt{x^2-4}$ doesn't exist.

side limit is right.

f. $\lim_{x \rightarrow 2} \sqrt{4-x^2} \Rightarrow \lim_{x \rightarrow 2} \sqrt{4-x^2} = 0$ only left side limit exist

$$g. \lim_{x \rightarrow -1} \sqrt{\sqrt{x^2 - 1} - x} \Rightarrow \lim_{x \rightarrow -1^-} \sqrt{\sqrt{x^2 - 1} - x}$$

does not exist

$$\lim_{x \rightarrow -1^-} \sqrt{\sqrt{x^2 - 1} - x} = \sqrt{0 + 1} = 1 \text{ (only left side limit is exist)}$$

Limit of piece wise – defined function

Sometimes a function is described by using different formulas in different parts of its domain.

Illustrative Examples

15. a. Let $f(x) = \begin{cases} x-1, & x < 3 \\ 3x-7, & x > 3 \end{cases}$ find $\lim_{x \rightarrow 3} f(x)$

Solution: $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} 3x-7 = 9-7 = 2 \leftarrow \text{right side limit}$

$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} x-1 = 3-1 = 2 \leftarrow \text{left side limit}$

$\therefore \lim_{x \rightarrow 3} f(x) = 2$ (since the one sided limits are equal)

b. Let $g(x) = \begin{cases} x^2-6, & x > 3 \\ 9-4x, & x < 3 \end{cases}$ then find $\lim_{x \rightarrow 3} g(x)$

Solution: $\lim_{x \rightarrow 3^+} g(x) = \lim_{x \rightarrow 3^+} x^2-6 = 9-6 = 3 \leftarrow \text{right side limit}$

$\lim_{x \rightarrow 3^-} g(x) = \lim_{x \rightarrow 3^-} (9-4x) = 9-12 = -3 \leftarrow \text{left side limit}$

$\therefore \lim_{x \rightarrow 3} g(x)$ does not exist since the right and the left side limits are not equal.

c. Let $k(x) = \begin{cases} \sqrt{2-x}, & x < 1 \\ 2x^2-x, & x > 1 \end{cases}$ then find $\lim_{x \rightarrow 1} k(x)$

Solution: $\lim_{x \rightarrow 1^+} k(x) = \lim_{x \rightarrow 1^+} 2x^2-x = 2-1 = 1$

$\lim_{x \rightarrow 1^-} k(x) = \lim_{x \rightarrow 1^-} \sqrt{2-x} = \sqrt{2-1} = 1$

d. Let $f(x) = \begin{cases} \frac{-2x-1}{2}, & \text{if } x < -1 \\ x^2 + 1, & \text{if } -1 < x \leq 2 \\ x+3 & \text{if } x > 2 \end{cases}$ then find:

i) $\lim_{x \rightarrow -1} f(x)$

ii) $\lim_{x \rightarrow 0} f(x)$

iii) $\lim_{x \rightarrow 2} f(x)$

Solution: i) $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} x^2 + 1 = 1 + 1 = 2$ and

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \frac{-2x-1}{2} = \frac{2-1}{2} = \frac{1}{2}$$

$\Rightarrow \lim_{x \rightarrow -1^+} f(x) \neq \lim_{x \rightarrow -1^-} f(x) \Rightarrow 2 \neq \frac{1}{2} \therefore \lim_{x \rightarrow -1} f(x)$ does not exist.

ii) $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 + 1 = 1$ and $\lim_{x \rightarrow 0^-} x^2 + 1 = 1$

$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) \therefore \lim_{x \rightarrow 0} f(x) = 1$

iii) $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x+3) = 2+3 = 5$ and

$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 + 1) = 4+1 = 5$

$\Rightarrow \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x), \therefore \lim_{x \rightarrow 2} f(x) = 5$

Difference between $\lim_{x \rightarrow a} f(x)$ and $f(a)$

- $\lim_{x \rightarrow a} f(x)$ is the value of $f(x)$ when x close to a but not at a .
- $f(a)$ is the value of $f(x)$ when exactly $x = a$. The following cases may arise.
 - i) $\lim_{x \rightarrow a} f(x)$ exist but $f(a)$ does not exist.
 - ii) $\lim_{x \rightarrow a} f(x)$ does not exist but $f(a)$ exist
 - iii) $\lim_{x \rightarrow a} f(x)$ and $f(a)$ both exist, but unequal
 - iv) $\lim_{x \rightarrow a} f(x)$ and $f(a)$ both exist, and equal
 - v) Neither $\lim_{x \rightarrow a} f(x)$ nor $f(a)$ exists

Limit involving Rational function

Indeterminate forms of type $\frac{0}{0}$

When both the numerator and the denominator of rational function $\frac{f(x)}{g(x)}$ have a zero at $x = a$, in which case $f(x)$ and $g(x)$ will each have a factor of $x - a$ and canceling this factor before evaluating the limit.

Illustrative Examples

16. Find the limit of the following.

a. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

b. $\lim_{x \rightarrow 4} \frac{x^2 - 3x - 4}{x^2 - 6x + 8}$

c. $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9}$

d. $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 3x + 2}$

e. $\lim_{x \rightarrow -7} \frac{x^2 - 49}{x^2 + 6x - 7}$

f. $\lim_{x \rightarrow -1} \frac{x^3 - 3x^2 - 4x}{x^2 + x}$

g. $\lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x^3 - 8}$

h. $\lim_{x \rightarrow -5} \frac{2x + 10}{x^2 + 6x + 5}$

Solution: Here

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \rightarrow 3} (x+3) = 3+3 = 6$$

$$\lim_{x \rightarrow 4} \frac{x^2 - 3x - 4}{x^2 - 6x + 8} = \lim_{x \rightarrow 4} \frac{(x+1)(x-4)}{(x-2)(x-4)} = \lim_{x \rightarrow 4} \frac{x+1}{x-2} = \frac{4+1}{4-2} = \frac{5}{2}$$

$$\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9} = \lim_{x \rightarrow 3} \frac{(x-3)(x+2)}{(x-3)(x+3)} = \lim_{x \rightarrow 3} \frac{x+2}{x+3} = \frac{3+2}{3+3} = \frac{5}{6}$$

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2 - 3x + 2} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x-1)} = \lim_{x \rightarrow 2} \frac{1}{x-1} = \frac{1}{2-1} = 1$$

$$\lim_{x \rightarrow -7} \frac{x^2 - 49}{x^2 - 6x - 7} = \lim_{x \rightarrow -7} \frac{(x+7)(x-7)}{(x+7)(x-1)} = \lim_{x \rightarrow -7} \frac{(x-7)}{x-1}$$

$$= \frac{-7-7}{-7-1} = \frac{-14}{-8} = \frac{7}{4}$$

$$\lim_{x \rightarrow -1} \frac{x^3 - 3x^2 - 4x}{x^2 + x} = \lim_{x \rightarrow -1} \frac{x(x+1)(x-4)}{(x)(x+1)}$$

$$= \lim_{x \rightarrow -1} (x-4) = -1-4 = -5$$

17. Find the limit of the following.

$$\text{a. } \lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{2}{x(x-1)(x-2)} \right] \quad \text{b. } \lim_{x \rightarrow 3} \left[\frac{1}{x-3} - \frac{3}{x(x^2 - 5x + 6)} \right]$$

$$\text{c. } \lim_{x \rightarrow 3} \left[\frac{1}{x-3} - \frac{6}{x^2 - 9} \right]$$

$$\text{d. } \lim_{x \rightarrow 0} \frac{1}{x} \left[\frac{2}{(2+x)^3} - \frac{1}{4} \right]$$

Solution: Here

$$\text{a. } \lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{2}{x(x-1)(x-2)} \right] = \lim_{x \rightarrow 2} \left[\frac{(x-2)(x+1)}{x(x-2)(x-1)} \right]$$

$$\begin{aligned} \text{b. } \lim_{x \rightarrow 3} \left[\frac{1}{x-3} - \frac{3}{x(x^2-5x+6)} \right] &= \lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{x(x-3)(x-2)} \\ &= \lim_{x \rightarrow 3} \frac{x+1}{x(x-2)} = \frac{4}{3} \end{aligned}$$

$$\text{c. } \lim_{x \rightarrow 3} \left[\frac{1}{x-3} - \frac{6}{x^2-9} \right] = \lim_{x \rightarrow 3} \frac{x-3}{(x-3)(x+3)} = \lim_{x \rightarrow 3} \frac{1}{x+3} = \frac{1}{6}$$

$$\begin{aligned} \text{d. } \lim_{x \rightarrow 0} \frac{1}{x} \left[\frac{2}{(2+x)^3} - \frac{1}{4} \right] &= \lim_{x \rightarrow 0} \frac{1}{x} \left[\frac{8 - (2+x)^3}{4(2+x)^3} \right] \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{1}{x} \left[\frac{8 - (x^3 + 6x^2 + 12x + 8)}{4(2+x)^3} \right] = \lim_{x \rightarrow 0} \frac{1}{x} \left[\frac{-x^3 - 6x^2 - 12x}{4(2+x)^3} \right] \end{aligned}$$

18. Find the limit of the following.

$$\text{a. } \lim_{x \rightarrow 2} \frac{\frac{1}{x+4} - \frac{1}{6}}{x-2}$$

$$\text{b. } \lim_{x \rightarrow 0} \frac{\frac{1}{x+3} - \frac{1}{3}}{x}$$

$$\text{c. } \lim_{x \rightarrow 7} \frac{\frac{1}{x} - \frac{1}{7}}{x-7}$$

$$\text{d. } \lim_{x \rightarrow a} \frac{x^{-3} - a^{-3}}{x^{-2} - a^{-2}}$$

Solution: Here

$$\text{a. } \lim_{x \rightarrow 2} \frac{\frac{1}{x+4} - \frac{1}{6}}{x-2} = \lim_{x \rightarrow 2} \frac{2-x}{(x-2)6(x+4)} = \lim_{x \rightarrow 2} \frac{-1}{6(x+4)} = \frac{-1}{36}$$

$$\text{b. } \lim_{x \rightarrow 0} \frac{\frac{1}{x+3} - \frac{1}{3}}{x} = \lim_{x \rightarrow 0} \frac{3-x-3}{x(x+3)3} = \lim_{x \rightarrow 0} \frac{-1}{3x+9} = \frac{-1}{9}$$

c. ← exercise left for you

$$\text{d. } \lim_{x \rightarrow a} \frac{x^{-3} - a^{-3}}{x^{-2} - a^{-2}} = \lim_{x \rightarrow a} \frac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{x^{\frac{1}{2}} - a^{\frac{1}{2}}} = \lim_{x \rightarrow a} \frac{(a^3 - x^3)(a^2 x^2)}{(a^3 x^3)(a^2 - x^2)}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{(a-x)(a^2+ax+x^2)}{(a-x)(a+x)(ax)} = \frac{a^2+a^2+a^2}{2a^3} = \frac{3a^2}{2a^3} = \frac{3}{2a}$$

• Limit involving greatest integer function

Recall that $f(x) = [x]$ denotes the greatest integer that does not exceed x .

19. Find the limit of the following function.

a) $\lim_{x \rightarrow \frac{1}{2}} [x]$ b) $\lim_{x \rightarrow \frac{4}{3}} [x]$ c) $\lim_{x \rightarrow \frac{4}{3}} [x]^2$ d) $\lim_{x \rightarrow \frac{4}{3}} [x^2]$
 e) $\lim_{x \rightarrow \frac{5}{4}} [x] + [-x]$ f) $\lim_{x \rightarrow 2} x - [x]$ g) $\lim_{x \rightarrow \frac{3}{2}} [x]^3$ h) $\lim_{x \rightarrow \frac{3}{2}} [x^3]$

Solution: a) $\lim_{x \rightarrow \frac{1}{2}} [x] = \left[\frac{1}{2} \right] = 0$

b) $\lim_{x \rightarrow \frac{4}{3}} [x] = \left[\frac{4}{3} \right] = 1$

c) $\lim_{x \rightarrow \frac{4}{3}} [x]^2 = \left[\frac{4}{3} \right]^2 = 1$

d) $\lim_{x \rightarrow \frac{4}{3}} [x^2] = \left[\frac{16}{9} \right] = 1$

e) $\lim_{x \rightarrow \frac{5}{4}} [x] + [-x] = \left[\frac{5}{4} \right] + \left[-\frac{5}{4} \right] = 1 + (-2) = -1$

f) $\lim_{x \rightarrow 2^+} x - [x] = 2 - 2 = 0$ and $\lim_{x \rightarrow 2^-} (x - [x]) = 2 - 1 = 1$ does not exist

g) $\lim_{x \rightarrow \frac{3}{2}} [x]^3 = \left[\frac{3}{2} \right]^3 = 1^3 = 1$

h) $\lim_{x \rightarrow \frac{3}{2}} [x^3] = \left[\frac{27}{8} \right] = 3$

Note:

If a is not an integer, then $\lim_{x \rightarrow a^+} [x] = \lim_{x \rightarrow a^-} [x] = [a]$

If a is an integer, then $\lim_{x \rightarrow a^+} [x] = a$ and $\lim_{x \rightarrow a^-} [x] = a - 1$

Therefore $\lim_{x \rightarrow a} [x]$ does not exist if a is an integer.



• Limits involving radicals

- In determinate forms of type $\frac{0}{0}$. When both the numerator

and the denominator of the expression $f(x) = \frac{N(x)}{D(x)}$ have a factor of

$x-a$. In this case $N(x)$ and $D(x)$ will each have a factor of $x-a$, after rationalizing the radical and canceling this factor may result in.

20. Find the limit of each of the following

a) $\lim_{x \rightarrow 1} \frac{\sqrt{x}-1}{x-1}$

b) $\lim_{x \rightarrow 2} \frac{\sqrt{x+2}-2}{x-2}$

c) $\lim_{x \rightarrow 0} \frac{\sqrt{x^2+4}-2}{x}$

d) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2}-1}{x^2}$

e) $\lim_{h \rightarrow 0} \frac{\sqrt{a+h}-\sqrt{a-h}}{h}$

f) $\lim_{x \rightarrow 3} \frac{\sqrt{2+3x}-\sqrt{2-3x}}{x}$

g) $\lim_{x \rightarrow 2} \frac{\sqrt{x}-\sqrt{2}}{x^2-2x}$

h) $\lim_{x \rightarrow 4} \frac{\sqrt{x-2}\sqrt{x-2}-2}{\sqrt{x}-2}$

i) $\lim_{x \rightarrow 3} \frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$

j) $\lim_{x \rightarrow -1} \frac{\sqrt{13+x}-3}{x+4}$

k) $\lim_{x \rightarrow 2^+} \frac{2-x}{\sqrt{4-4x+x^2}}$

l) $\lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\sqrt{9+h}} - \frac{1}{3} \right)$

m) $\lim_{x \rightarrow 4^+} \frac{x-4}{\sqrt{2x-4}-\sqrt{3x-8}}$

n) $\lim_{x \rightarrow 1} \frac{\sqrt{x-1}+\sqrt{x-1}}{\sqrt{x^2-1}}$

Solution:

$$a) \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1}$$

$$= \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \lim_{x \rightarrow 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \rightarrow 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

$$b) \lim_{x \rightarrow 2} \frac{\sqrt{x+2} - 2}{x - 2} = \lim_{x \rightarrow 2} \frac{\sqrt{x+2} - 2}{x - 2} \cdot \frac{\sqrt{x+2} + 2}{\sqrt{x+2} + 2} = \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(\sqrt{x+2} + 2)}$$

$$= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x+2} + 2} = \frac{1}{4}$$

$$c) \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 4} - 2}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 4} - 2}{x} \cdot \frac{\sqrt{x^2 + 4} + 2}{\sqrt{x^2 + 4} + 2}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 + 4 - 4}{x(\sqrt{x^2 + 4} + 2)} = \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x^2 + 4} + 2)} = 0$$

$$d) \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x^2} = \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x^2} \cdot \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}$$

$$= \lim_{x \rightarrow 0} \frac{2x^2}{x^2(\sqrt{1+x^2} + \sqrt{1-x^2})} = \frac{2}{2} = 1$$

$$e) \lim_{h \rightarrow 0} \frac{\sqrt{a+h} - \sqrt{a-h}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{a+h} - \sqrt{a-h}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a-h}}{\sqrt{a+h} + \sqrt{a-h}}$$

$$= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{a+h} + \sqrt{a-h})} = \frac{2}{2\sqrt{a}}$$

$$f) \lim_{x \rightarrow 0} \frac{\sqrt{2+3x} - \sqrt{2-3x}}{x} \cdot \frac{\sqrt{2+3x} + \sqrt{2-3x}}{\sqrt{2+3x} + \sqrt{2-3x}} = \lim_{x \rightarrow 0} \frac{6x}{x(\sqrt{2+3x} + \sqrt{2-3x})} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$g) \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x^2 - 2x} \cdot \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}} = \lim_{x \rightarrow 2} \frac{x - 2}{x(x - 2)(\sqrt{x} + \sqrt{2})} = \frac{1}{4\sqrt{2}}$$

$$h) \lim_{x \rightarrow 4} \frac{\sqrt{(\sqrt{x} - 1)^2} - 1}{\sqrt{x} - 2} = \lim_{x \rightarrow 4} \frac{|\sqrt{x} - 1| - 1}{\sqrt{x} - 2} = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{\sqrt{x} - 2} = 1$$

$$\begin{aligned} \text{o) } \lim_{x \rightarrow 3} \frac{\sqrt{3x-3}}{\sqrt{2x-4}-\sqrt{2}} &= \lim_{x \rightarrow 3} \frac{\sqrt{3x-3}}{\sqrt{2x-4}-\sqrt{2}} \left(\frac{\sqrt{2x-4}+\sqrt{2}}{\sqrt{2x-4}+\sqrt{2}} \right) \left(\frac{\sqrt{3x+3}}{\sqrt{3x+3}} \right) \\ &= \lim_{x \rightarrow 3} \frac{(3x-9)(\sqrt{2x-4}+\sqrt{2})}{(2x-4-2)(\sqrt{3x+3})} = \lim_{x \rightarrow 3} \frac{3(x-3)}{2(x-3)} \cdot \lim_{x \rightarrow 3} \frac{\sqrt{2x-4}+\sqrt{2}}{\sqrt{3x+3}} = \frac{3}{2} \cdot \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{2} \end{aligned}$$

$$\begin{aligned} \text{p) } \lim_{x \rightarrow -4} \frac{\sqrt{13+x}-3}{x+4} &= \frac{3-\sqrt{13+x}}{(x+4)(\sqrt{13+x})3} \\ &= \lim_{x \rightarrow -4} \frac{3-\sqrt{13+x}}{(x+4)(3)(\sqrt{13+x})} \cdot \left(\frac{3+\sqrt{13+x}}{3+\sqrt{13+x}} \right) \\ &= \lim_{x \rightarrow -4} \frac{9-(13+x)}{(x+4)(3)(\sqrt{13+x})(3+\sqrt{13+x})} \\ &= \lim_{x \rightarrow -4} \frac{-(x+4)}{x+4} \cdot \lim_{x \rightarrow -4} \frac{1}{3\sqrt{13+x}} \cdot \lim_{x \rightarrow -4} \frac{1}{(3+\sqrt{13+x})} = (-1) \cdot \frac{1}{9} \cdot \frac{1}{6} = -\frac{1}{54} \end{aligned}$$

$$\text{q) } \lim_{x \rightarrow 2^+} \frac{2-x}{\sqrt{4-4x+x^2}} = \lim_{x \rightarrow 2^+} \frac{2-x}{\sqrt{(2-x)^2}} = \lim_{x \rightarrow 2^+} \frac{2-x}{|2-x|} = \lim_{x \rightarrow 2^+} \frac{2-x}{-(2-x)} = -1$$

$$\text{However, } \lim_{x \rightarrow 2^-} \frac{2-x}{|2-x|} = \lim_{x \rightarrow 2^-} \frac{2-x}{2-x} = 1$$

$$\begin{aligned} \text{r) } \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\sqrt{9+h}} - \frac{1}{3} \right) &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{3-\sqrt{9+h}}{3\sqrt{9+h}} \right] \left[\frac{3+\sqrt{9+h}}{3+\sqrt{9+h}} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{9-(9+h)}{(3\sqrt{9+h})(3-\sqrt{9+h})} \right] \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} \cdot \lim_{h \rightarrow 0} \frac{1}{(3\sqrt{9+h})(3+\sqrt{9+h})} = (-1) \cdot \frac{1}{9} \cdot \frac{1}{6} = -\frac{1}{54} \end{aligned}$$

$$\begin{aligned} \text{s) } \lim_{x \rightarrow 4^+} \frac{x-4}{\sqrt{2x-4}-\sqrt{3x-8}} &= \lim_{x \rightarrow 4^+} \frac{x-4}{\sqrt{2x-4}-\sqrt{3x-8}} \cdot \frac{\sqrt{2x-4}+\sqrt{3x-8}}{\sqrt{2x-4}+\sqrt{3x-8}} \\ &= \lim_{x \rightarrow 4^+} \frac{(x-4)(\sqrt{2x-4}+\sqrt{3x-8})}{2x-4-(3x-8)} = \lim_{x \rightarrow 4^+} \frac{(x-4)(\sqrt{2x-4}+\sqrt{3x-8})}{4-x} = -4 \end{aligned}$$

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$$1) \lim_{x \rightarrow 1} \frac{\sqrt{x-1} + \sqrt{x} - 1}{\sqrt{x^2 - 1}} = \lim_{x \rightarrow 1} \left(\frac{\sqrt{x-1}}{\sqrt{x^2 - 1}} + \frac{\sqrt{x} - 1}{\sqrt{x^2 - 1}} \right),$$

$$\lim_{x \rightarrow 1} \left(\frac{\sqrt{x-1}}{\sqrt{x-1}\sqrt{x+1}} + \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x-1}\sqrt{x+1} \cdot \sqrt{x+1}} \right)$$

$$= \lim_{x \rightarrow 1} \left[\frac{1}{\sqrt{x+1}} + \frac{x-1}{\sqrt{x+1}\sqrt{x+1}} \right] = \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$$

21. Find the value of k so that $\lim_{x \rightarrow 3^-} f(x)$ exist;

$$\text{where } f(x) = \begin{cases} 2x - 7, & \text{for } x < 3 \\ -3x + k^2, & \text{for } x \geq 3 \end{cases}$$

Solution:

$$\text{Here } \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (2x - 7) = 6 - 7 = -1 \text{ and}$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (-3x + k^2) = -9 + k^2$$

$$\text{Since } \lim_{x \rightarrow 3} f(x) \text{ exists; } \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x)$$

$$\Rightarrow k^2 - 9 = -1 \Rightarrow k^2 = -1 + 9 = 8$$

$$\therefore k = \pm\sqrt{8} = \pm 2\sqrt{2}$$

22. Evaluate: $\lim_{x \rightarrow 1} \frac{x^2 - 1}{\log x}$

Solution:

$$\text{Recall that } \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \text{ and } \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{\log x} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{\log[1+(x-1)]} \leftarrow \text{divide both side by } x-1$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{x+1}{\frac{\log[1+(x-1)]}{x-1}} = \frac{\lim_{x \rightarrow 1} x+1}{\lim_{x \rightarrow 1} \frac{\log[1+(x-1)]}{x-1}} = \frac{1+1}{1} = 2$$

23. If $\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} g(x)$, then does it follow that $f(x) = g(x)$ for all x ?

Answer: NO

24. Let $f(x) = \frac{x}{x-3}$ and $g(x) = \frac{x^2-9}{x^2+4x}$ then find
- a) $\lim_{x \rightarrow 3} (f \cdot g)(x)$ b) $\lim_{x \rightarrow 0} (f \cdot g)(x)$

Solution: a) $\lim_{x \rightarrow 3} (f \cdot g)(x) = \lim_{x \rightarrow 3} \frac{x}{x-3} \cdot \frac{x^2-9}{x^2+4x} = \lim_{x \rightarrow 3} \frac{x}{x-3} \cdot \frac{(x-3)(x+3)}{(x)(x+4)}$

$$= \lim_{x \rightarrow 3} \frac{x+3}{x+4} = \frac{3+3}{3+4} = \frac{6}{7}$$

b) $\lim_{x \rightarrow 0} (f \cdot g)(x) = \lim_{x \rightarrow 0} \frac{x+3}{x+4} = \frac{3}{4}$

Limit involving absolute value

The absolute value function $y = |x|$ is defined piece wisely by the formula. $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Here, we investigate the limit value near the given value of the domain.

Illustrative Examples

21. Find the limit (left and right hand limit). Does the two side limit exist?

a. $\lim_{x \rightarrow 0} \frac{x}{|x|}$

b. $\lim_{x \rightarrow 0} \frac{3x + |x|}{3x - |x|}$

c. $\lim_{x \rightarrow 2} \frac{|x-4|}{x-2}$

d. $\lim_{x \rightarrow -3} \frac{x^2-9}{|x+3|}$

e. $\lim_{x \rightarrow 3} \frac{x^2|x^2-9|}{x-3}$

f. $\lim_{x \rightarrow 0} \frac{x-5|x|}{x}$

g. $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$

h. $\lim_{x \rightarrow -2} \frac{x^3+8}{|x+2|}$

i. $\lim_{x \rightarrow 3} \frac{|x^2-x-6|}{x-3}$

Solution: Here

a. $\lim_{x \rightarrow 0} \frac{x}{|x|} \Rightarrow \lim_{x \rightarrow 0^+} \frac{x}{x} = 1$ and $\lim_{x \rightarrow 0^-} \left(\frac{x}{x} \right) = -1 \therefore \lim_{x \rightarrow 0} \frac{x}{|x|}$ does not exist, since one side limit are not equal.

$$\lim_{x \rightarrow 0} \frac{3x - |x|}{3x + |x|} \Rightarrow \lim_{x \rightarrow 0^+} \frac{3x + x}{3x - x} = \frac{4x}{2x} = 2 \text{ and } \lim_{x \rightarrow 0^-} \frac{3x - |x|}{3x + |x|} = \frac{2x}{4x} = \frac{1}{2}$$

$$\therefore \lim_{x \rightarrow 0} \frac{3x + |x|}{3x - |x|} \text{ does not exist.}$$

$$c. \lim_{x \rightarrow 2} \frac{|x^2 - 4|}{x - 2} \Rightarrow \lim_{x \rightarrow 2^+} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(x - 2)(x + 2)}{x - 2} = 2 + 2 = 4$$

$$\text{Also, } \lim_{x \rightarrow 2^-} \frac{-(x - 2)(x + 2)}{x - 2} = - \lim_{x \rightarrow 2^-} (x + 2) = -(2 + 2) = -4$$

$$d. \lim_{x \rightarrow -3} \frac{x^2 - 9}{|x + 3|} \Rightarrow \lim_{x \rightarrow -3^+} \frac{(x - 3)(x + 3)}{(x + 3)} = \lim_{x \rightarrow -3^+} (x - 3) = -6 \text{ also}$$

$$\lim_{x \rightarrow -3^-} \frac{(x - 3)(x + 3)}{-(x + 3)} = \lim_{x \rightarrow -3^-} -(x - 3) = 6 \quad \lim_{x \rightarrow -3} \frac{x^2 - 9}{|x + 3|}$$

does not exist.

$$e. \lim_{x \rightarrow 3} \frac{x^2 |x^2 - 9|}{x - 3} \Rightarrow \lim_{x \rightarrow 3^+} \frac{x^2 (x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3^+} x^2 (x + 3) = 54$$

$$\text{Also, } \lim_{x \rightarrow 3^-} \frac{-(x^2)(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3^-} (-x^2)(x + 3) = -54$$

$$\therefore \lim_{x \rightarrow 3} \frac{x^2 (x^2 - 9)}{x - 3} \text{ does not exist.}$$

f, g, h ← exercise left for you

$$i. \lim_{x \rightarrow 3} \frac{|x^2 - x - 6|}{x - 3} \Rightarrow \lim_{x \rightarrow 3^+} \frac{(x - 3)(x + 2)}{(x - 3)} = \lim_{x \rightarrow 3^+} (x + 2) = 5$$

$$\text{Also, } \lim_{x \rightarrow 3^-} \frac{-(x - 3)(x + 2)}{x - 3} = - \lim_{x \rightarrow 3^-} (x + 2) = -(3 + 2) = -5$$

$$\therefore \lim_{x \rightarrow 3} \frac{|x^2 - x - 6|}{x - 3} \text{ does not exist.}$$

22. Find the one side limit of the following.

$$a. \lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{x^2 - 4x + 4}}$$

$$b. \lim_{x \rightarrow 3} \frac{x + 3}{\sqrt{x^2 + 6x + 9}}$$

$$c. \lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{\sqrt{x^2 + 2x + 1}}$$

$$d. \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{\sqrt{x - 4}\sqrt{x + 4}}$$

$$e. \lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x^2 - 8x + 16}}$$

Solution: Recall that: $\sqrt{(x+a)^2} = |x+a|$

$$a. \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x^2 - 4x + 4}} \Rightarrow \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{(x-2)^2}} = \lim_{x \rightarrow 2} \frac{x-2}{|x-2|}$$

$$\Rightarrow \lim_{x \rightarrow 2^+} \frac{x-2}{x-2} = 1 \text{ and } \lim_{x \rightarrow 2^-} \frac{x-2}{x-2} = -1$$

$$b. \lim_{x \rightarrow 3} \frac{x+3}{\sqrt{x^2 + 6x + 9}} \Rightarrow \lim_{x \rightarrow 3} \frac{x+3}{\sqrt{(x+3)^2}} \Rightarrow \lim_{x \rightarrow 3} \frac{x+3}{|x+3|}$$

$$\therefore \lim_{x \rightarrow 3^+} \frac{x+3}{x+3} = 1 \text{ and } \lim_{x \rightarrow 3^-} \frac{(x+3)}{-(x+3)} = -1$$

$$c. \lim_{x \rightarrow -1} \frac{x^2 - 2x - 3}{\sqrt{(x+1)^2}} = \lim_{x \rightarrow -1} \frac{(x+1)(x-3)}{|x+1|} = \lim_{x \rightarrow -1^+} (x-3) = -4 \text{ also}$$

$$\lim_{x \rightarrow -1^-} \frac{(x+1)(x-3)}{-(x+1)} = \lim_{x \rightarrow -1^-} -(x-3) = -(-4) = 4$$

$$d. \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{\sqrt{x - 4}\sqrt{x + 4}} \Rightarrow \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{\sqrt{(\sqrt{x} - 2)^2}} \Rightarrow \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{|\sqrt{x} - 2|}$$

$$\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{\sqrt{x} - 2} = 1 \text{ and } \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{-(\sqrt{x} - 2)} = -1$$

$$e. \lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x^2 - 8x + 16}} \leftarrow \text{Exercise left for you.}$$

Limit involving quotient difference function

The expression:

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$$i) \frac{f(x+h) - f(x)}{h}, h \neq 0$$

$$ii) \frac{f(x) - f(x_0)}{x - x_0}, x \neq x_0 \text{ is called the quotient difference of } f(x).$$

Illustrative Examples

23. a) Let $f(x) = x^3 - x$, then find $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$.

Solution: $\lim_{x \rightarrow 2} \frac{x^3 - x - 6}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 3)}{x - 2} = 11$

b) Let $f(x) = \sqrt{4 - x^2}$, then find $\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1}$.

Solution: $\lim_{x \rightarrow 1} \frac{\sqrt{4 - x^2} - \sqrt{3}}{x - 1} \leftarrow \text{since it is } \frac{0}{0} \text{ form}$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\sqrt{4 - x^2} - \sqrt{3}}{x - 1} \cdot \frac{\sqrt{4 - x^2} + \sqrt{3}}{\sqrt{4 - x^2} + \sqrt{3}} = \lim_{x \rightarrow 1} \frac{(4 - x^2 - 3)}{(x - 1)(\sqrt{4 - x^2} + \sqrt{3})}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{(1 - x)(1 + x)}{(x - 1)(\sqrt{4 - x} + \sqrt{3})} = \frac{-(1)(2)}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

c) Let $g(x) = \begin{cases} x^2 - 4 & \text{if } x \geq 2 \\ -x^2 + 4 & \text{if } x < 2 \end{cases}$ then find $\lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2}$.

Solution: $\lim_{x \rightarrow 2^+} \frac{x^2 - 4 - 0}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(x - 2)(x + 2)}{x - 2} = 2 + 2 = 4$ and

$$\lim_{x \rightarrow 2^-} \frac{g(x) - g(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{4 - x^2}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(2 - x)(x + 2)}{x - 2} = -4$$

$\therefore \lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2}$ does not exist.

d) Let $f(x) = x^n$, then find $\lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$.

Solution:

$$\Rightarrow \lim_{x \rightarrow t} \frac{t^n - x^n}{t - x} = \lim_{t \rightarrow x} \frac{(t - x) [t^{n-1} + xt^{n-2} + x^2t^{n-3} + \dots + x^{n-2}t + x^{n-1}]}{t - x}$$

$$= \lim_{t \rightarrow x} [t^{n-1} + xt^{n-2} + x^2t^{n-3} + \dots + x^{n-2}t + x^{n-1}]$$

$$= x^{n-1} + x^{n-1} + x^{n-1} + \dots + x^{n-1} = nx^{n-1}$$

e) Find $\lim_{t \rightarrow x} \frac{t^3 - x^3}{t - x} \leftarrow$ Exercise left for you (Ans. $3x^2$)

Limit of exponential function**A) The number 'e' expressed as a limit**

• The number 'e' as a limit:

i) $\lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$

ii) $\lim_{x \rightarrow \pm\infty} \left(1 + \frac{n}{x}\right)^x = e^n$

iii) $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$

iv) $\lim_{x \rightarrow 0} (1+nx)^{\frac{1}{x}} = e^n$

Illustrative Examples

24. Evaluate the limit of the following.

a. $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{3x}$

b. $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^{3x}$

c. $\lim_{x \rightarrow \infty} \left(\frac{x+4}{x-1}\right)^{x+4}$

d. $\lim_{x \rightarrow \infty} \left(\frac{3x+4}{3x+1}\right)^{2x+1}$

e. $\lim_{x \rightarrow \infty} \left(\frac{3x}{3x+4}\right)^x$

f. $\lim_{x \rightarrow \infty} \left(\frac{x}{x+3}\right)^{8-5x}$

Solution: a) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{3x} = \lim_{x \rightarrow \infty} \left(\left(1 + \frac{1}{x}\right)^x \right)^3 = e^3$

b) $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^{3x} = \lim_{x \rightarrow \infty} \left(\left(1 + \frac{-1}{x}\right)^x \right)^3 = (e^{-1})^3 = e^{-3}$

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$$c) \lim_{x \rightarrow \infty} \left(\frac{x+4}{x-1} \right)^{x+4} = \lim_{x \rightarrow \infty} \left(\frac{1 + \frac{4}{x}}{1 - \frac{1}{x}} \right)^{x+4} = \frac{e^4}{e^{-1}} = e^5$$

$$d) \lim_{x \rightarrow \infty} \left(\frac{3x+4}{3x+1} \right)^{2x+1} = \lim_{x \rightarrow \infty} \left(\frac{1 + \frac{4}{3x}}{1 + \frac{1}{3x}} \right)^{2x+1}$$

$$= \lim_{x \rightarrow \infty} \left[\frac{1 + \frac{4}{3x}}{1 + \frac{1}{3x}} \right]^{2x} \cdot \lim_{x \rightarrow \infty} \left[\frac{1 + \frac{4}{3x}}{1 + \frac{1}{3x}} \right]^1 = \left(\frac{e^{\frac{4}{3}}}{e^{\frac{1}{3}}} \right)^2 \cdot 1 = \frac{e^{\frac{8}{3}}}{e^{\frac{2}{3}}} = e^{\frac{8-2}{3}} = e^2$$

$$e) \lim_{x \rightarrow \infty} \left(\frac{3x}{3x-4} \right)^x = \lim_{x \rightarrow \infty} \left[\frac{1}{1 - \frac{4}{3x}} \right]^x = \frac{1}{\frac{1}{e^{\frac{4}{3}}}} = e^{\frac{4}{3}}$$

$$f) \lim_{x \rightarrow \infty} \left[\frac{x}{x+3} \right]^{8-5x} = \lim_{x \rightarrow \infty} \left[\frac{1}{1 + \frac{3}{x}} \right]^{5x} \cdot \lim_{x \rightarrow \infty} \left[\frac{1}{1 + \frac{3}{x}} \right]^8$$

$$= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{3}{x} \right)^{-1} \right]^{5x} \cdot 1 = \lim_{x \rightarrow \infty} \left[\left(1 + \frac{3}{x} \right)^x \right]^{-5} \cdot 1 = (e^3)^{-5} = e^{-15}$$

29. If $\lim_{x \rightarrow \infty} \left[\frac{x+k}{x-k} \right]^x = 4$, then find value of k .

$$\text{Solution: } \lim_{x \rightarrow \infty} \left[\frac{x+k}{x-k} \right]^x = 4 \Rightarrow \lim_{x \rightarrow \infty} \left[\frac{1 + \frac{k}{x}}{1 - \frac{k}{x}} \right]^x = 4$$

$$\Rightarrow \frac{e^k}{e^{-k}} = 4 \Rightarrow e^{2k} = 4 \Rightarrow 2k = \ln 4 \Rightarrow k = \frac{\ln 4}{2}$$

$$\therefore k = \ln 2$$

26. Find the limit of the following.

a. $\lim_{x \rightarrow 0} (1+3x)^{\frac{1}{x}}$ b. $\lim_{x \rightarrow 0} (1-4x)^{\frac{1}{x}}$ c. $\lim_{x \rightarrow 0} (6x+1)^{\frac{1}{2x}}$

Solution: Recall that: i) $\lim_{x \rightarrow 0} [1+x]^{\frac{1}{x}} = e$

i) $\lim_{x \rightarrow 0} [1+cx]^{\frac{1}{x}} = e^c$, for any, $c \in \mathbb{R}$

a) $\lim_{x \rightarrow 0} [1+3x]^{\frac{1}{x}} = e^3$ b) $\lim_{x \rightarrow 0} (1-4x)^{\frac{1}{x}} = e^{-4}$

c) $\lim_{x \rightarrow 0} [1+6x]^{\frac{1}{2x}} = (e^6)^{\frac{1}{2}} = e^3$

Limit of $\frac{a^x - 1}{x}$

i) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$

ii) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \ln e = 1$

To proof: Let $a^x - 1 = h \Rightarrow a^x = h + 1 \Rightarrow x = \log_a^{(h+1)}$. As $x \rightarrow 0 \Rightarrow h \rightarrow 0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{a^x - 1}{x} \Rightarrow \lim_{h \rightarrow 0} \frac{h}{\log_a^{(h+1)}} \Rightarrow \frac{1}{\lim_{h \rightarrow 0} \frac{1}{h} \log_a^{(h+1)}}$$

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$$\lim_{h \rightarrow 0} \left(\log(h+1)^{\frac{1}{h}} \right) = \frac{1}{\log a} \lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}} = \frac{1}{\log_e e} = \ln a$$

$$\therefore \lim_{x \rightarrow 0} \left[\frac{a^x - 1}{x} \right] = \ln a$$

27. Find the limit of the following.

a. $\lim_{x \rightarrow 0} \frac{2^x - 1}{x}$

b. $\lim_{x \rightarrow 0} \left[\frac{3^x - 1}{4x} \right]$

c. $\lim_{x \rightarrow 0} \frac{e^x - 1}{3x}$

d. $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x}$

d. $\lim_{x \rightarrow 0} \frac{4^x - 3^x}{x}$

Solution: a) $\lim_{x \rightarrow 0} \frac{2^x - 1}{x} = \ln 2$

b) $\lim_{x \rightarrow 0} \frac{3^x - 1}{4x} = \frac{1}{4} \lim_{x \rightarrow 0} \frac{3^x - 1}{x} = \frac{1}{4} \ln 3 = \frac{\ln 3}{4}$

c) $\lim_{x \rightarrow 0} \frac{e^x - 1}{3x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \frac{1}{3} \ln e = \frac{1}{3}$

d) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} = \lim_{x \rightarrow 0} \frac{e^x - 1 + 1 - e^{-x}}{x}$
 $= \lim_{x \rightarrow 0} \frac{e^x - 1}{x} + \lim_{x \rightarrow 0} \frac{e^x - 1}{x} \cdot \frac{1}{e^x} = \ln e + \ln e = 1 + 1 = 2$

e) $\lim_{x \rightarrow 0} \frac{4^x - 3^x}{x} = \lim_{x \rightarrow 0} \frac{\left(\frac{4}{3}\right)^x - 1}{x} = \ln \left(\frac{4}{3}\right) = \ln 4 - \ln 3$

Method II, $\lim_{x \rightarrow 0} \frac{(4^x - 1) - (3^x - 1)}{x} = \lim_{x \rightarrow 0} \frac{4^x - 1}{x} - \lim_{x \rightarrow 0} \frac{3^x - 1}{x} = \ln 4 - \ln 3$

Limit involving $\frac{\sin x}{x}$

$$\text{i) } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\text{ii) } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\text{iii) } \lim_{x \rightarrow 0} \frac{\sin kx}{x} = k$$

$$\text{iv) } \lim_{x \rightarrow 0} \frac{\sin kx}{mx} = \frac{k}{m}$$

Illustrative Examples

28. Evaluate each of the following limits.

$$\text{a. } \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x}$$

$$\text{b. } \lim_{x \rightarrow 0} \frac{\sin 2x}{\tan 5x}$$

$$\text{c. } \lim_{x \rightarrow 0} \frac{x}{\sin 4x}$$

Solution: Here

$$\text{a) } \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x} \Rightarrow \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{x}}{\frac{\sin 5x}{x}} = \frac{3}{5} \leftarrow \text{Multiply each by } \frac{1}{x}$$

$$\text{b) } \lim_{x \rightarrow 0} \frac{\sin 2x}{\tan 5x} \Rightarrow \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{x}}{\frac{\tan 5x}{x}} = \frac{2}{5}$$

$$\text{c) } \lim_{x \rightarrow 0} \frac{x}{\sin 4x} \Rightarrow \lim_{x \rightarrow 0} \frac{\frac{x}{x}}{\frac{\sin 4x}{x}} = \frac{1}{4}$$

$$\text{Note: } \lim_{x \rightarrow a} \frac{\sin(x-a)}{x-a} = 1 \leftarrow \text{form } \frac{0}{0}$$

$$\lim_{x \rightarrow -a} \frac{\sin(x+a)}{x+a} = 1 \leftarrow \text{still } \frac{0}{0}$$

29. Evaluate each of the limit.

$$\text{a. } \lim_{x \rightarrow 4} \frac{\sin(x-4)}{x^2-16}$$

$$\text{b. } \lim_{x \rightarrow -1} \frac{\sin(x^2+3x+2)}{x^3+1}$$

$$\text{c. } \lim_{x \rightarrow 0} \frac{\sin x}{x^3+2x}$$

$$\text{d. } \lim_{x \rightarrow 0} \frac{\sin(2x+4)}{x^2+5x+6}$$

Solution: Here, four of them have the form $\frac{0}{0}$.

$$a) \lim_{x \rightarrow 4} \frac{\sin(x-4)}{x^2-16} = \lim_{x \rightarrow 4} \frac{\sin(x-4)}{x-4} \cdot \lim_{x \rightarrow 4} \frac{1}{x+4} = 1 \cdot \frac{1}{4+4} = \frac{1}{8}$$

$$b) \lim_{x \rightarrow -1} \frac{\sin(x^2+3x+2)}{x^3+1} \Rightarrow \lim_{x \rightarrow -1} \frac{\sin[(x+2)(x+1)]}{(x+1)(x^2-x+1)}$$

$$\Rightarrow \lim_{x \rightarrow -1} \left[\frac{\sin[(x+2)(x+1)]}{(x+1)(x+2)} \cdot \frac{(x+1)(x+2)}{(x+1)(x^2-x+1)} \right]$$

$$= \lim_{x \rightarrow -1} \frac{\sin[(x+2)(x+1)]}{(x+1)(x+2)} \cdot \lim_{x \rightarrow -1} \frac{(x+2)}{x^2-x+1} = (1) \left(\frac{1}{3} \right) = \frac{1}{3}$$

$$c) \lim_{x \rightarrow 0} \frac{\sin x}{x^3+2x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{x^2+2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

30. Find the limit value of the following.

$$a) \lim_{x \rightarrow 0} \frac{\cos\left(x + \frac{\pi}{2}\right)}{x}$$

$$b) \lim_{x \rightarrow 1} \frac{\sin \pi x}{x-1}$$

$$c) \lim_{x \rightarrow \pi} \frac{\sin \pi - x}{\sin x}$$

$$d) \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{x^2 - \frac{\pi^2}{4}}$$

$$e) \lim_{x \rightarrow \pi} \frac{\sin \pi x}{x^2 - \pi^2}$$

$$f) \lim_{x \rightarrow 0} \frac{\sin^2 3x}{x - x\sqrt{1-x}}$$

Solution: Here, from (a - f) has $\frac{0}{0}$ form.

$$a) \lim_{x \rightarrow 0} \frac{\cos\left(x + \frac{\pi}{2}\right)}{x} = \lim_{x \rightarrow 0} \frac{-\sin x}{x} = -1$$

$$b) \text{ Let } y = x - 1 \Rightarrow x = y + 1, \text{ as } x \rightarrow 1, y \rightarrow 0$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\sin \pi x}{x-1} \Rightarrow \lim_{y \rightarrow 0} \frac{\sin(\pi(y+1))}{y} \Rightarrow \lim_{y \rightarrow 0} \frac{\sin(\pi y + \pi)}{y}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{[\sin \pi y \cos \pi + \cos \pi y \sin \pi]}{y}$$

$$\text{since } \sin \pi = 0 \text{ and } \cos \pi = -1 \Rightarrow \lim_{y \rightarrow 0} \frac{-\sin \pi y}{y} = -\pi$$

$$\text{c) Let } y = \pi - x, x = \pi - y, \text{ as } x \rightarrow \pi, y \rightarrow 0$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{\pi - x}{\sin x} \Rightarrow \lim_{y \rightarrow 0} \frac{y}{\sin(\pi - y)}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{y}{\sin \pi \cos y - \cos \pi \sin y} \lim_{y \rightarrow 0} \frac{y}{\sin y} = 1$$

$$\text{d) } \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{x^2 - \frac{\pi^2}{2}} \Rightarrow \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\left(x - \frac{\pi}{2}\right)\left(x + \frac{\pi}{2}\right)}$$

$$\text{Let } x - \frac{\pi}{2} = y \Rightarrow x = y + \frac{\pi}{2}, x \rightarrow \frac{\pi}{2}, y \rightarrow 0$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{\cos\left(y + \frac{\pi}{2}\right)}{y(y + \pi)} \Rightarrow \lim_{y \rightarrow 0} \frac{-\sin y}{y} \cdot \lim_{y \rightarrow 0} \frac{1}{y + \pi} = -\frac{1}{\pi}$$

$$\text{e) } \lim_{x \rightarrow \pi} \frac{\sin x}{x^2 - \pi^2} \leftarrow \text{Exercise left for you, Ans. } -\frac{1}{2\pi}$$

$$\text{f) } \lim_{x \rightarrow 0} \frac{\sin^2 3x}{x - x\sqrt{1-x}} \Rightarrow \lim_{x \rightarrow 0} \frac{\sin^2 3x}{x(1 - \sqrt{1-x})}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin^2 3x}{x(1 - \sqrt{1-x})} \cdot \left(\frac{1 + \sqrt{1-x}}{1 + \sqrt{1-x}}\right) \Rightarrow \lim_{x \rightarrow 0} \frac{\sin^2 3x}{x(1 - (1-x))} \cdot \lim_{x \rightarrow 0} (1 + \sqrt{1-x})$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{x}\right)^2 \cdot (2) = (3)^2 \cdot (2) = 18$$

35. Find the limit value of the following.

$$\text{a. } \lim_{x \rightarrow 0} 2x \csc(3x)$$

$$\text{c. } \lim_{x \rightarrow \infty} x^2 \sin \frac{1}{x^2}$$

$$\text{b. } \lim_{x \rightarrow 0} x \cot 3x$$

$$\text{d. } \lim_{x \rightarrow 0} x \tan \frac{1}{x}$$

Solution: Here

$$a) \lim_{x \rightarrow 0} (2x) \csc(3x) \Rightarrow \lim_{x \rightarrow 0} \frac{2x}{\sin 3x} = \frac{2}{3}$$

$$b) \lim_{x \rightarrow 0} x \cot 3x \Rightarrow \lim_{x \rightarrow 0} \frac{x}{\tan 3x} = \frac{1}{3}$$

$$c) \text{ Let } y = \frac{1}{x} \Rightarrow x^2 = \frac{1}{y}, \text{ as } x \rightarrow \infty, y \rightarrow 0 \\ \Rightarrow \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

$$d) \text{ Let } y = \frac{1}{x} \Rightarrow y = \frac{1}{x}, \text{ as } x \rightarrow \infty,$$

$$y \rightarrow 0 \Rightarrow \lim_{x \rightarrow \infty} x \tan\left(\frac{1}{x}\right) = \lim_{y \rightarrow 0} \frac{\tan y}{y} = 1$$

$$\text{Note: i) } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \quad \text{ii) } \lim_{x \rightarrow 0} \frac{1 - \cos(ax)}{x^2} = \frac{a^2}{2}$$

To check (ii) it has $\frac{0}{0}$ form.

$$\lim_{x \rightarrow 0} \frac{1 - \cos(ax)}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos ax}{x^2} \cdot \frac{1 + \cos ax}{1 + \cos ax}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos^2 ax}{x^2} \cdot \lim_{x \rightarrow 0} \frac{1}{1 + \cos ax}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin^2 ax}{x^2} \cdot \frac{1}{2} = \left(\lim_{x \rightarrow 0} \frac{\sin ax}{x} \right)^2 \cdot \frac{1}{2} = \frac{a^2}{2}$$

Illustrative Examples

Find the limit value of the following

$$\text{a. } \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{x^2}$$

$$\text{c. } \lim_{x \rightarrow \pi} \frac{\tan^2 x}{1 + \sec x}$$

$$\text{b. } \lim_{x \rightarrow 0} \frac{\cos 4x - 1}{x^2}$$

$$\text{d. } \lim_{x \rightarrow 0} \frac{x}{\sqrt{1 - \cos x}}$$

Solution: All has $\frac{0}{0}$ form:

$$\text{a. } \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{x^2} = \frac{3^2}{2} = \frac{9}{2} \leftarrow \text{from (ii)}$$

$$\text{b. } \lim_{x \rightarrow 0} \frac{\cos(4x) - 1}{x^2} \Rightarrow \lim_{x \rightarrow 0} \frac{(\cos 4x - 1)(\cos 4x + 1)}{x^2 (\cos 4x + 1)}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\cos^2 4x - 1}{x^2} \cdot \frac{1}{(\cos 4x + 1)}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin^2 4x}{x^2} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos 4x + 1}$$

$$= - \left(\lim_{x \rightarrow 0} \frac{\sin 4x}{x} \right)^2 \cdot \frac{1}{2} = \frac{-16}{2} = -8$$

$$\text{c. } \lim_{x \rightarrow \pi} \frac{\tan^2 x}{1 + \sec x} \Rightarrow \lim_{x \rightarrow \pi} \frac{\sec^2 x - 1}{1 + \sec x}$$

$$\Rightarrow \lim_{x \rightarrow \pi} \frac{(\sec x - 1)(\sec x + 1)}{1 + \sec x} = -1 - 1 = -2$$

$$\text{d. } \lim_{x \rightarrow 0} \frac{x}{\sqrt{1 - \cos x}} \Rightarrow \lim_{x \rightarrow 0} \frac{x}{\sqrt{1 - \cos x}} \cdot \frac{\sqrt{1 + \cos x}}{\sqrt{1 + \cos x}}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x}{\sqrt{1 - \cos^2 x}} \cdot \sqrt{1 + \cos x}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x}{\sqrt{\sin^2 x}} \cdot \lim_{x \rightarrow 0} \sqrt{1 + \cos x} = \lim_{x \rightarrow 0} \frac{x}{|\sin x|} \cdot \sqrt{2}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{x}{\sin x} \cdot \sqrt{2} = \sqrt{2},$$

also $\lim_{x \rightarrow 0^+} \frac{x}{-\sin x} = -\sqrt{2}$, since $|\sin x| = \begin{cases} \sin x, & x \geq 0 \\ -\sin x, & x < 0 \end{cases}$

$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1-\cos x}}$ doesn't exist because the one side are not equal.

$\lim_{x \rightarrow 0^-} \frac{x}{\sqrt{1-\cos x}}$ doesn't exist because the one side are not equal.

$$\lim_{x \rightarrow 0^+} \frac{x}{\sqrt{1-\cos x}} = -\sqrt{2} \text{ and } \lim_{x \rightarrow 0^-} \frac{x}{\sqrt{1-\cos x}} = \sqrt{2}$$

37. Find the limit of the following.

a. $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}}$

b. $\lim_{x \rightarrow 0^+} \frac{\cos \sqrt{x}}{\sin x}$

c. $\lim_{x \rightarrow 0^+} \cos \frac{1}{x}$

Solution: Here

a. $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \lim_{x \rightarrow 0^+} \sqrt{x} \frac{\sin x}{x} = 0(1) = 0$

b. $\lim_{x \rightarrow 0^+} \frac{\cos \sqrt{x}}{\sin x} = \frac{\cos(0^+)}{\sin 0^+} = \frac{1}{0^+} = \infty$

c. $\lim_{x \rightarrow 0^+} \cos \frac{1}{x} = \cos \left(\frac{1}{0^+} \right) = \cos(\infty)$ it is oscillating between -1

and 1.

38. Find the limit of the following:

a. $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$

c. $\lim_{x \rightarrow 0} \frac{x \sin 4x}{1 - \cos 4x}$

b. $\lim_{x \rightarrow 2} \frac{\sin(x-2) + x^2 - 4}{x-2}$

d. $\lim_{x \rightarrow \infty} x \sin \left(\frac{\pi}{x} \right)$

e. $\lim_{x \rightarrow 0} \frac{\sin^3(2x)}{\sin(4x^3)}$

f. $\lim_{x \rightarrow 0} \frac{\sin \left(\frac{x}{\pi} \right)}{\tan x}$

Solution: Here, all has $\frac{0}{0}$ form:

$$\begin{aligned}
 \text{a. } \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} \\
 &\Rightarrow \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{x^3 \cos x} \Rightarrow \lim_{x \rightarrow 0} \frac{(\sin x)(1 - \cos x)(1 + \cos x)}{x^3 \cos x(1 + \cos x)} \\
 &\Rightarrow \lim_{x \rightarrow 0} \frac{(\sin x)(1 - \cos^2 x)}{x^3 \cos x(1 + \cos x)} \Rightarrow \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^3} \cdot \frac{1}{\cos x(1 + \cos x)} = \frac{1}{2} \\
 \text{b. } \lim_{x \rightarrow 2} \frac{\sin(x-2) + (x^2 - 4)}{x-2} &\Rightarrow \lim_{x \rightarrow 2} \frac{\sin(x-2)}{x-2} + \lim_{x \rightarrow 2} \frac{x^2 - 4}{x-2} \\
 &\Rightarrow (1) \lim_{x \rightarrow 2} (x+2) = (1)(4) = 4 \\
 \text{c. } \lim_{x \rightarrow 0} \frac{x \sin 4x}{1 - \cos 4x} &\Rightarrow \lim_{x \rightarrow 0} \frac{x \sin 4x(1 + \cos 4x)}{(1 - \cos 4x)(1 + \cos 4x)} \\
 &\Rightarrow \lim_{x \rightarrow 0} \frac{x \sin 4x(1 + \cos 4x)}{1 - \cos^2 4x} \Rightarrow \lim_{x \rightarrow 0} \frac{x \sin 4x}{\sin^2 4x} \cdot \lim_{x \rightarrow 0} (1 + \cos 4x) \\
 &\Rightarrow \lim_{x \rightarrow 0} \frac{x}{\sin 4x} (1 + 1) = \frac{1}{4} \cdot (2) = \frac{1}{2} \\
 \text{d. } \lim_{x \rightarrow 0} x \sin\left(\frac{\pi}{x}\right) &\leftarrow \text{exercise left for you} \\
 \text{e. } \lim_{x \rightarrow 0} \frac{\sin^3 2x}{\sin 4x^3} &= \frac{\lim_{x \rightarrow 0} 8 \left(\frac{\sin 2x}{2x}\right)^3}{\lim_{x \rightarrow 0} 4 \frac{\sin 4x^3}{4x^3}} = \frac{8}{4} = 2
 \end{aligned}$$

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$$\lim_{x \rightarrow 0} \frac{\sin\left(\frac{x}{\pi}\right)}{\tan x} = \lim_{x \rightarrow 0} \frac{\frac{\sin\left(\frac{x}{\pi}\right)}{\frac{x}{\pi}}}{\frac{\tan x}{x}} = \frac{1}{\pi}$$

39. Find the limit value of the following:

a. $\lim_{x \rightarrow 0} (\ln \sin 2x - \ln(x^2 + 3x))$

b. $\lim_{x \rightarrow 0} \sqrt{\frac{\sin 9x}{x}}$

c. $\lim_{x \rightarrow 0} \frac{\sin^{-1} 3x}{2x}$

d. $\lim_{x \rightarrow 0} \frac{3x - \sin 4x}{x + \sin 5x}$

e. $\lim_{x \rightarrow 0} \frac{\cos(5x) - \cos(3x)}{x^2}$

Solution: Here

$$\begin{aligned} \text{a. } \lim_{x \rightarrow 0} [\ln(\sin 2x) - \ln(x^2 + 3x)] &= \lim_{x \rightarrow 0} \ln \left(\frac{\sin 2x}{x^2 + 3x} \right) \\ &= \ln \left(\lim_{x \rightarrow 0} \frac{\sin 2x}{x} \cdot \frac{1}{x+3} \right) = \ln \left(\frac{2}{3} \right) = \ln 2 - \ln 3 \end{aligned}$$

$$\text{b. } \lim_{x \rightarrow 0} \sqrt{\frac{\sin 9x}{x}} = \sqrt{\lim_{x \rightarrow 0} \frac{\sin 9x}{x}} = \sqrt{9} = 3$$

$$\text{c. Let } y = \sin^{-1} 3x \Rightarrow \sin y = 3x \Rightarrow \sin y = 3x \Rightarrow x = \frac{\sin y}{3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin^{-1}(3x)}{2x} = \lim_{y \rightarrow 0} \frac{y}{2 \left(\frac{\sin y}{3} \right)} = \frac{3}{2}$$

$$\text{d. } \lim_{x \rightarrow 0} \frac{3x - \sin 4x}{x + \sin 5x} = \lim_{x \rightarrow 0} \frac{\frac{3x - \sin 4x}{x}}{\frac{x + \sin 5x}{x}} = \frac{3 - 4}{1 + 5} = \frac{-1}{6}$$

$$\begin{aligned} \text{c. } \lim_{x \rightarrow 0} \frac{\cos(5x) - \cos(3x)}{x^2} &= \lim_{x \rightarrow 0} \frac{\cos(5x) - 1 + 1 - \cos 3x}{x^2} \\ &\Rightarrow \lim_{x \rightarrow 0} \frac{\cos(5x) - 1}{x^2} + \lim_{x \rightarrow 0} \frac{(1 - \cos(3x))}{x^2} = \frac{-25}{2} + \frac{9}{2} = \frac{-16}{2} = -8 \end{aligned}$$

What is the value of $\lim_{x \rightarrow 0} \frac{\cos x}{x}$

$$\lim_{x \rightarrow 0^+} \frac{\cos x}{x} = \frac{1}{0^+} = \infty, \text{ and } \lim_{x \rightarrow 0^-} \frac{\cos x}{x} = \frac{1}{0^-} = -\infty \leftarrow 0 \text{ indicates very small negative number near to } 0.$$

$\therefore \lim_{x \rightarrow 0} \frac{\cos x}{x}$ doesn't exist, since it is not finite.

40. Evaluate:

$$\text{a. } \lim_{x \rightarrow \infty} \frac{\sin 3x}{x} \quad \text{b. } \lim_{x \rightarrow \infty} \frac{\cos x}{x} \quad \text{c. } \lim_{x \rightarrow \infty} \frac{\tan x}{x}$$

Solution: Here

$$\text{a. } \lim_{x \rightarrow \infty} \frac{\sin 3x}{x} = 0 \quad \text{b. } \lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0$$

$$\text{c. } \lim_{x \rightarrow \infty} \frac{\tan x}{x} \leftarrow \text{exercise left for you}$$

Non-existence of limit

The two condition for nonexistence of the limits.

i) If the one side limit are not equal, then the two sided limit doesn't exist.

For example, $\lim_{x \rightarrow 0} \frac{|x|}{x}$ doesn't exist b/c $\lim_{x \rightarrow 0^+} \frac{x}{x} = 1$ and $\lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$

ii) Sometimes one sided or two sided limits will fail to exist because the value of the function increase or decrease without bound, this called infinite limit.

Infinite limits and vertical asymptote

Infinite limits provide best tool and symbol for describing the behavior of function whose value become arbitrarily large positive or negative

Definition (Informal)

Infinite limits: If the value of $f(x)$ increase or decrease without bound as x close to 'a' from the right or left, then we expressed

$$\lim_{x \rightarrow a} f(x) = \pm \infty$$

Two sided infinite limit

If $\lim_{x \rightarrow a^+} f(x) = \infty$ and $\lim_{x \rightarrow a^-} f(x) = \infty$

$\therefore \lim_{x \rightarrow a} f(x) = \infty \leftarrow$ two sided infinite limit ($f(x)$ increase

without bound). **Example:** $\lim_{x \rightarrow 0} \frac{1}{|x|} = \infty$

If $\lim_{x \rightarrow a^+} f(x) = -\infty$ and $\lim_{x \rightarrow a^-} f(x) = -\infty$

$\therefore \lim_{x \rightarrow a} f(x) = -\infty \leftarrow$ two sided infinite limit ($f(x)$ decrease

without bound). **Example:** $\lim_{x \rightarrow 2} \frac{2-x}{(x-2)^3} = -\infty$

One Sided infinite limit

If $\lim_{x \rightarrow a^+} f(x) = -\infty$ and $\lim_{x \rightarrow a^-} f(x) = +\infty$ or

$\lim_{x \rightarrow a^+} f(x) = +\infty$ and $\lim_{x \rightarrow a^-} f(x) = -\infty$ then it is called one sided infinite limit.

Note: The phrase " $f(x)$ approaches ∞ " is best to say that " $f(x)$ approaches the unapproachable or $f(x)$ goes to infinity."

Vertical asymptote

A line $x = a$ is a vertical asymptote of the graph of function.

$y = f(x)$ if either $\lim_{x \rightarrow a^+} f(x) = \pm \infty$ or $\lim_{x \rightarrow a^-} f(x) = \pm \infty$

Illustrative Examples

1. Evaluate and discuss the one sided, the two sided limit value and vertical asymptote of the graph.

$$\lim_{x \rightarrow 2} \frac{-1}{x-2}$$

$$\text{b. } \lim_{x \rightarrow 2} \frac{-1}{(x-2)^2}$$

$$\text{c. } \lim_{x \rightarrow -3} \frac{2x+5}{x+3}$$

d. $\lim_{x \rightarrow 1} \frac{x+4}{x-1}$

e. $\lim_{x \rightarrow 3} \frac{2x^2 - 4x + 5}{3-x}$

f. $\lim_{x \rightarrow -3} \frac{x+1}{x^2 + 2x - 3}$

g. $\lim_{x \rightarrow 4} \frac{3-x}{x^2 - 2x - 8}$

h. $\lim_{x \rightarrow 1} \frac{x^2 - x}{(x-1)^2}$

i. $\lim_{x \rightarrow -1} \frac{x^2 - 2}{(x+1)^2}$

j. $\lim_{x \rightarrow 4} \frac{4-x}{(x-4)^3}$

k. $\lim_{x \rightarrow 3} \frac{x-3}{(x-3)^3}$

Solution: Take 0^+ means slightly greater than 0 (very small positive number)

0^- means slightly less than 0 (very small negative number) and

$$\frac{1}{0^+} = \frac{1}{\text{very small positive number}} = \infty$$

$$\frac{1}{0^-} = \frac{1}{\text{very small negative number}} = -\infty$$

a. $\lim_{x \rightarrow 2} \frac{-1}{x-2} = \frac{-1}{2^+ - 2} = \frac{-1}{0^+} = -\infty \leftarrow \text{right side infinite limit}$

$$\lim_{x \rightarrow 2^-} \frac{-1}{x-2} = \frac{-1}{2^- - 2} = \frac{-1}{0^-} = \infty \leftarrow \text{left sided infinite limit}$$

$\therefore x = 2$ is vertical asymptote

b. $\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = \frac{1}{0^+} = \infty \leftarrow \text{two sided infinite limit. Limit from each side is } \infty. \therefore \text{The line } x = 2 \text{ is vertical asymptote.}$

c. $\lim_{x \rightarrow -3^+} \frac{2x+5}{x+3} = \frac{-6+5}{0^+} = \frac{-1}{0^+} = -\infty$

$$\lim_{x \rightarrow -3^-} \frac{2x+5}{x+3} = \frac{-6+5}{0^-} = \frac{-1}{0^-} = \infty$$

$\therefore x = -3$ is vertical asymptote

d. $\lim_{x \rightarrow 1^+} \frac{x+4}{x-1} = \frac{1+4}{1^+ - 1} = \frac{5}{0^+} = +\infty, \lim_{x \rightarrow 1^-} \frac{x+4}{x-1} = \frac{1+4}{1^- - 1} = \frac{5}{0^-} = -\infty$

$\therefore x = 1 \leftarrow \text{vertical asymptote.}$

e. $\lim_{x \rightarrow 3^+} \frac{2x^2 - 4x + 5}{3-x} = \frac{18-12+5}{3-3^+} = \frac{11}{0^-} = -\infty$

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$$\lim_{x \rightarrow 3^-} \frac{2x^2 - 4x + 5}{3 - x} = \frac{11}{0^+} = \infty \therefore x = 3 \leftarrow \text{vertical asymptote.}$$

$$\lim_{x \rightarrow 3^+} \frac{x+1}{x^2 + 2x - 3} = \lim_{x \rightarrow 3^+} \frac{x+1}{(x-1)(x+3)} = \frac{-2}{(-4)0^+} = \infty$$

$$\lim_{x \rightarrow -3} \frac{x+1}{(x-1)(x+3)} = \frac{-2}{(-4)(0^-)} = -\infty \therefore \text{The line } x = -3 \text{ and } x = 1$$

are vertical asymptote.

$$g. \lim_{x \rightarrow 4^-} \frac{3-x}{x^2 - 2x - 8} = \lim_{x \rightarrow 4^-} \frac{3-x}{(x-4)(x+2)} = \frac{3-4}{(0^+)(6)} = \frac{-1}{0^+} = -\infty$$

$$\lim_{x \rightarrow 4^+} \frac{3-x}{(x-4)(x+2)} = \frac{-1}{(0^-)(6)} = \frac{-1}{0^-} = \infty$$

$$h. \lim_{x \rightarrow 1^-} \frac{x^2 - x}{(x-1)^2} = \lim_{x \rightarrow 1^-} \frac{x(x-1)}{(x-1)^2} = \lim_{x \rightarrow 1^-} \frac{x}{x-1} = \frac{1}{0^+} = \infty$$

$$\lim_{x \rightarrow 1^+} \frac{x^2 - x}{(x-1)^2} = \lim_{x \rightarrow 1^+} \frac{x}{x-1} = \frac{1}{0^-} = -\infty$$

$$i. \lim_{x \rightarrow -1^+} \frac{x^2 - 2}{(x+1)^2} = \frac{1-2}{0^+} = \frac{-1}{0^+} = -\infty, \quad \lim_{x \rightarrow -1^-} \frac{x^2 - 2}{(x+1)^2} = \frac{1-2}{0^-} = \frac{-1}{0^-} = \infty$$

$$j. \lim_{x \rightarrow -1} \frac{x^2 - 2}{(x+1)^2} = -\infty \leftarrow \text{two sided infinite limit. (limit form each side is } -\infty)$$

$$k. \lim_{x \rightarrow 4^-} \frac{4-x}{(x-4)^3} = \lim_{x \rightarrow 4^-} \frac{-1}{(x-4)^2} = \frac{-1}{0^+} = -\infty$$

$$l. \lim_{x \rightarrow 4^+} \frac{4-x}{(x-4)^3} = \lim_{x \rightarrow 4^+} \frac{-1}{(x-4)^2} = \frac{-1}{0^+} = -\infty \therefore \lim_{x \rightarrow 4} \frac{4-x}{(x-4)^3} = -\infty$$

is two sided infinite limit. Limit form each side is $-\infty$.

$$m. \lim_{x \rightarrow 3} \frac{x-3}{(x-3)^3} \leftarrow \text{exercise left for you}$$



Infinite limit involving logarithm

42. Find the limit of the following.

a. $\lim_{x \rightarrow 1^-} \ln(x-1)$

c. $\lim_{x \rightarrow 2} \log_3(x^2 - 4)$

b. $\lim_{x \rightarrow 1^+} \ln(x-1)^2$

d. $\lim_{x \rightarrow 3^+} \log_{\frac{1}{2}}(x-3)$

Solution:

a. $\lim_{x \rightarrow 1^-} \ln(x-1) = -\infty$

b. $\lim_{x \rightarrow 1^+} \ln(x-1)^2 = -\infty$ and $\lim_{x \rightarrow 1^-} \ln(x-1)^2 = -\infty$

$\therefore \lim_{x \rightarrow 1} \ln(x-1)^2 = -\infty$

c. $\lim_{x \rightarrow 2} \log_3(x^2 - 4) = -\infty$

d. $\lim_{x \rightarrow 3^+} \log_{\frac{1}{2}}(x-3) = \infty$

43. Find the limit of the following.

a. $\lim_{x \rightarrow \frac{\pi}{2}} \tan x$

c. $\lim_{x \rightarrow \pi} \cot x$

b. $\lim_{x \rightarrow \frac{\pi}{2}} \sec x$

d. $\lim_{x \rightarrow \pi} \csc x$

Solution:

a. $\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x = -\infty$ and $\lim_{x \rightarrow \frac{\pi}{2}^+} \tan x = \infty$

b. $\lim_{x \rightarrow \pi^-} \cot x = \infty$ and $\lim_{x \rightarrow \pi^+} \cot x = -\infty$

c. $\lim_{x \rightarrow \frac{\pi}{2}^+} \sec x = -\infty$ and $\lim_{x \rightarrow \frac{\pi}{2}^-} \sec x = \infty$

d. $\lim_{x \rightarrow \pi^-} \csc x = -\infty$ and $\lim_{x \rightarrow \pi^+} \csc x = \infty$

Limit at infinite and horizontal asymptote

The behavior of a function toward the extreme value of its domain is sometimes called **end behavior**. Here we will use limits to investigate the end behavior of the function as $x \rightarrow \infty$. If the value of $f(x)$ get closer and closer to a number L as x increase without bound, then write $\lim_{x \rightarrow \infty} f(x) = L$.

If the values of $f(x)$ get closer and closer to a number L as x decrease without bound, then write:

$$\lim_{x \rightarrow -\infty} f(x) = L$$

Definition: A line $y = L$ is called a horizontal asymptote of the graph of function if $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$.

Illustrative Examples

44. Find the limit and horizontal asymptote.

a. $\lim_{x \rightarrow \infty} \frac{5x^2 + 6x - 3}{4x^2 + 7}$

d. $\lim_{x \rightarrow \infty} \frac{\sin x - x + 3}{\cos x + 2x}$

b. $\lim_{x \rightarrow \infty} \frac{-2x^3 + 4x + 5}{3x^3 + 6x^2 - 7}$

e. $\lim_{x \rightarrow \infty} \frac{\cos x}{3x}$

c. $\lim_{x \rightarrow \infty} \frac{4x + 6}{3x - 10}$

Solution: We can divide the numerator and denominator by highest power of x .

a. $\lim_{x \rightarrow \infty} \frac{5x^2 + 6x - 3}{4x^2 + 7} = \lim_{x \rightarrow \infty} \frac{5 + \frac{6}{x} - \frac{3}{x^2}}{4 + \frac{7}{x^2}}$ divide numerator

and denominator by x^2 , $= \frac{5 + 0 - 0}{4 + 0} = \frac{5}{4}$ ← horizontal asymptote

b. $\lim_{x \rightarrow \infty} \frac{-2x^3 + 4x + 5}{3x^3 + 6x^2 - 7}$
denominator by x^3

Divide the numerator and

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{-2 + \frac{4}{x^2} + \frac{5}{x^3}}{3 + \frac{6}{x} - \frac{7}{x^3}} = \frac{-2 + 0 + 0}{3 + 0 - 0} = \frac{-2}{3} \leftarrow \text{horizontal asymptote}$$

c. $\lim_{x \rightarrow \infty} \frac{4x+6}{3x-10}$ Divide the numerator and denominator by x

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{4 + \frac{6}{x}}{3 - \frac{10}{x}} = \frac{4+0}{3-0} = \frac{4}{3} \leftarrow \text{horizontal asymptote}$$

d. $\lim_{x \rightarrow \infty} \frac{\sin x - x + 3}{\cos x + 2x} = \lim_{x \rightarrow \infty} \frac{\frac{\sin x}{x} - 1 + \frac{3}{x}}{\frac{\cos x}{x} + 2} = \frac{0 - 1 + 0}{0 + 2} = \frac{-1}{2}$

$$\therefore y = \frac{-1}{2} \leftarrow \text{horizontal asymptote}$$

e. $\lim_{x \rightarrow \infty} \frac{\cos x}{3x} = 0, \therefore y = 0$ is H.A.

45. Show that the graph of f has two horizontal asymptotes and determine.

a. $f(x) = \frac{x|x|}{x^2+1}$

c. $f(x) = \frac{\sqrt{3x^2-6}}{x+4}$

b. $f(x) = \frac{\sqrt{2+4x^2}}{x}$

d. $f(x) = \sqrt{\frac{1-3x}{3-x}}$

Solution: Recall that, $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

a. $\lim_{x \rightarrow \infty} \frac{x|x|}{x^2+1} = \lim_{x \rightarrow \infty} \frac{x^2}{x^2+1} = \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x^2}} = 1 \therefore y = 1$ is H.A.

$$\lim_{x \rightarrow \infty} \frac{x|x|}{x^2+1} = \lim_{x \rightarrow \infty} \frac{-x^2}{x^2+1} = \frac{-1}{\lim_{x \rightarrow \infty} 1 + \frac{1}{x^2}} = -1 \therefore y = -1 \text{ is H.A.}$$

$$1. f(x) = \frac{\sqrt{2+4x^2}}{x} = \frac{\sqrt{4x^2+2}}{x} = \frac{\sqrt{x^2\left(4+\frac{2}{x^2}\right)}}{x} = \frac{|x|\sqrt{4+\frac{2}{x^2}}}{x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{|x|\sqrt{4+\frac{2}{x^2}}}{2} = 2, \therefore y = 2 \text{ is H.A.}$$

$$\text{Also, } \lim_{x \rightarrow -\infty} \frac{|x|\sqrt{4+\frac{2}{x^2}}}{x} = -2, \therefore y = -2 \text{ is H.A.}$$

$$2. f(x) = \frac{\sqrt{3x^2-6}}{x+4} = \frac{\sqrt{x^2\left(3-\frac{6}{x^2}\right)}}{x\left(1+\frac{4}{x}\right)} = \frac{|x|\sqrt{3-\frac{6}{x^2}}}{x\left(1+\frac{4}{x}\right)}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{|x|\sqrt{3-\frac{6}{x^2}}}{x\left(1+\frac{4}{x}\right)} = \sqrt{3}, \therefore y = \sqrt{3} \text{ is H.A.}$$

$$\text{Also, } \lim_{x \rightarrow -\infty} \frac{|x|\sqrt{3-\frac{6}{x^2}}}{x\left(1+\frac{4}{x}\right)} = -\sqrt{3}, \text{ since } x < 0, \frac{|x|}{x} = -1$$

$\therefore y = -\sqrt{3} \leftarrow$ Horizontal asymptote.

$$3. \lim_{x \rightarrow \infty} \sqrt{\frac{1-3x}{3-x}} = \lim_{x \rightarrow \infty} \sqrt{\frac{\frac{1}{x}-3}{\frac{3}{x}-1}} = \sqrt{\frac{0-3}{0-1}} = \sqrt{3}$$

46. Find the limits of the following.

a. $\lim_{x \rightarrow \infty} \sqrt{x^2 + 5x} - x$

c. $\lim_{x \rightarrow \infty} \sqrt{x^2 + ax} - x$

b. $\lim_{x \rightarrow \infty} \sqrt{x^4 + 3x^2} - x^2$

d. $\lim_{x \rightarrow \infty} \sqrt[3]{\frac{3+4x-6x^2}{1+8x^2}}$

Solution: Here

$$\begin{aligned} \text{a. } \lim_{x \rightarrow \infty} \sqrt{x^2 + 5x} - x &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 5x} - x)(\sqrt{x^2 + 5x} + x)}{\sqrt{x^2 + 5x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 + 5x - x^2}{\sqrt{x^2 + 5x} + x} = \lim_{x \rightarrow \infty} \frac{5x}{\sqrt{x^2 \left(1 + \frac{5}{x}\right)} + x} = \lim_{x \rightarrow \infty} \frac{5x}{x \left(\sqrt{1 + \frac{5}{x}} + 1\right)} = \frac{5}{2} \end{aligned}$$

b. $\lim_{x \rightarrow \infty} \sqrt{x^2 + ax} - x \leftarrow \text{Exercise left for you. (Ans. } \frac{a}{2} \text{)}$

$$\begin{aligned} \text{c. } \lim_{x \rightarrow \infty} \sqrt{x^4 + 3x^2} - x^2 &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^4 + 3x^2} - x^2)(\sqrt{x^4 + 3x^2} + x^2)}{\sqrt{x^4 + 3x^2} + x^2} \\ &= \lim_{x \rightarrow \infty} \frac{x^4 + 3x^2 - x^4}{\sqrt{x^4 + 3x^2} + x^2} = \lim_{x \rightarrow \infty} \frac{3x^2}{\sqrt{x^4 \left(1 + \frac{3}{x^2}\right)} + x^2} = \lim_{x \rightarrow \infty} \frac{3x^2}{x^2 \left(\sqrt{1 + \frac{3}{x^2}} + 1\right)} = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{d. } \lim_{x \rightarrow \infty} \sqrt[3]{\frac{3+4x-6x^2}{1+8x^2}} &= \sqrt[3]{\lim_{x \rightarrow \infty} \frac{\left(\frac{3}{x^2} + \frac{4}{x} - 6\right)}{\left(\frac{1}{x^2} + 8\right)}} = \sqrt[3]{\frac{-6}{8}} = \frac{\sqrt[3]{-6}}{2} \end{aligned}$$

Continuity of function

The term continuous means without break. The graph of continuous function has no gap, no jump, no hole.

Continuity of a function at a point

Definition: A function f is said to be continuous at $x = a$ provided the following conditions are satisfied:

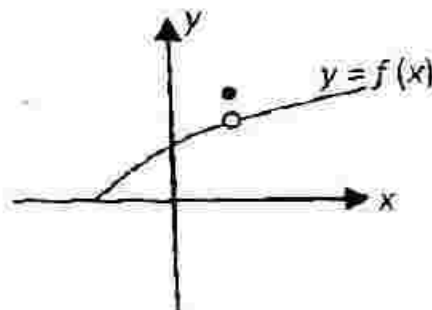
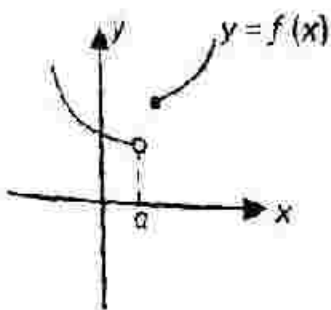
- i) $f(a)$ is defined
- ii) $\lim_{x \rightarrow a} f(x)$ exist
- iii) $\lim_{x \rightarrow a} f(x) = f(a)$

Illustrative Examples

47. Discuss the continuity of each function.

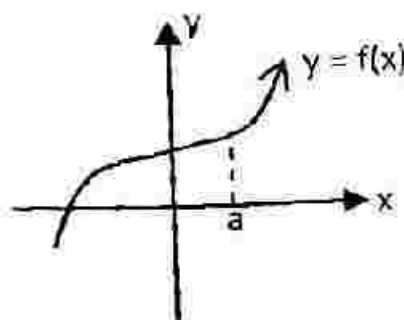
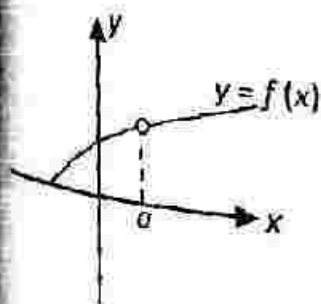
a.

B.



c.

d.



Solution: Here

a. $\lim_{x \rightarrow a} f(x)$ does not exist, violating 2nd condition. $\therefore f$ is not continuous at $x = a$.

b. i) $\lim_{x \rightarrow a} f(x)$ exist ii) $f(a)$ is defined, but

iii) $\lim_{x \rightarrow a} f(x) \neq f(a) \leftarrow$ violating 3rd condition $\therefore y = f(x)$ is not con. at $x = a$

- c. $\lim_{x \rightarrow a} f(x)$ exist, but $f(a)$ is not defined, thus violating 1st condition. $\therefore y = f(x)$ is not cont. at $x = a$.
- d. $\lim_{x \rightarrow a} f(x)$ exist and $\lim_{x \rightarrow a} f(x) = f(a) \therefore y = f(x)$ is cont. at $x = a$.

48. Determine whether the following functions are continuous at indicated point.

a. $f(x) = \begin{cases} x^2 - 4x + 6, & x < 2 \\ -x^2 + 4x - 2, & x \geq 2 \end{cases}$ at $x = 2$

b. $g(x) = \begin{cases} \frac{x^2 - 7x + 12}{x - 3}, & x \neq 3 \\ -1, & \text{if } x = 3, \end{cases}$ at $x = 3$

c. $k(x) = \begin{cases} \frac{(x+2)|x+4|}{x+4}, & x \neq -4 \\ -2, & \text{if } x = -4 \end{cases}$

d. $h(x) = \begin{cases} 3x + 5, & x \leq -1 \\ 2x^3 + 4, & x > -1 \end{cases}$ at $x = -1$

e. $R(x) = [x] - x$ at $x = 2$

f. $f(x) = \begin{cases} \frac{\sin 2x}{x}, & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$ at $x = 0$

g. $f(x) = \begin{cases} (1+3x)^{\frac{1}{x}} & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$ at $x = 0$

Solution: Here, use the three continuity conditions.

- a. To show continuity at $x = 2$. i) $f(2) = -4 + 8 - 2 = 2$, and
 $\lim_{x \rightarrow 2} f(x) = 2 \Rightarrow \lim_{x \rightarrow 2} f(x) = f(2)$ Therefore, f is cont.
 at $x = 2$

b. i) $g(3) = -1$ and $\lim_{x \rightarrow 3} g(x) = \lim_{x \rightarrow 3} \frac{(x-3)(x-4)}{x-3} = -1$

$\Rightarrow \lim_{x \rightarrow 3} g(x) = g(3)$ Therefore, g is cont. at $x = 3$

c. $k(-4) = -2$ and $\lim_{k \rightarrow -4^+} k(x) \Rightarrow \lim_{k \rightarrow -4^-} k(x)$

$\Rightarrow \lim_{k \rightarrow -4^+} k(x) \Rightarrow \lim_{k \rightarrow -4^+} (x+2) = -2$ and

$\lim_{k \rightarrow -4^-} k(x) = \lim_{k \rightarrow -4^-} -(x+2) = -(-2) = 2$

$\lim_{k \rightarrow -4^+} k(x) \neq \lim_{k \rightarrow -4^-} k(x) \Rightarrow \lim_{k \rightarrow -4} k(x)$ doesn't exist.

$\therefore k$ is not cont. at $x = -4$.

d. $h(-1) = -3 + 5$ and

$\lim_{h \rightarrow -1} h(x) = 2 \Rightarrow \lim_{h \rightarrow -1} h(x) = h(-1) \therefore h$ is cont. at $x = -1$.

e. $R(2) = [2] - 2 = 0$ and

$\lim_{x \rightarrow 2} R(x) \Rightarrow \lim_{x \rightarrow 2^+} R(x) = \lim_{x \rightarrow 2^+} [2^+] - 2 = 0$.

$\Rightarrow \lim_{x \rightarrow 2^+} R(x) = \lim_{x \rightarrow 2^+} [2^+] - 2 = 0$, also

$\lim_{x \rightarrow 2^-} R(x) = \lim_{x \rightarrow 2^-} [2^-] - 2 = 1 - 2 = -1 \Rightarrow \lim_{x \rightarrow 2} R(x)$ does not exist.

$\therefore R$ is not cont. at $x = 2$.

f. $f(0) = 2$ and

$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2 \Rightarrow \lim_{x \rightarrow 0} f(x) = f(0)$

$\therefore f$ is cont. at $x = 0$, therefore, f is cont. at $x = 0$.

g. $f(0) = 3$ and

$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (1 + 3x)^{\frac{1}{x}} = e^3 \Rightarrow \lim_{x \rightarrow 0} f(x) \neq f(0)$

$\therefore f$ is cont. not at $x = 0$.

One side continuity

- A function f is continuous from the right at $x = a$ provided that $\lim_{x \rightarrow a^+} f(x) = f(a)$.
- A function f is continuous from the left at $x = b$ provided that $\lim_{x \rightarrow b^-} f(x) = f(b)$.

Illustrative Examples

49. Examine the continuity of the following:

- a. $f(x) = \sqrt{2-x}$, at $x=2$ b. $f(x) = \sqrt{9-x^2}$, at $x=-3$
 d. $f(x) = \sqrt{4-x^2}$, at $x=2, -2$ c. $f(x) = [x]$ at $x=4$

Solution:

- a. domain = $\{x: 2-x \geq 0\} = (-\infty, 2] \Rightarrow \lim_{x \rightarrow 2^-} f(x) = f(2) = 0$
 $\therefore f$ is cont. from the left at $x = 2$
 b. domain = $\{x: 9-x^2 \geq 0\} = [-3, 3]$
 $\Rightarrow \lim_{x \rightarrow -3^+} f(x) = f(-3) = 0$ and $\Rightarrow \lim_{x \rightarrow 3^-} f(x) = f(3) = 0$
 c. $\lim_{x \rightarrow 4^+} f(x) = 4$ and $\lim_{x \rightarrow 4^-} f(x) = 3$ and $f(4) = 4$
 $\Rightarrow \lim_{x \rightarrow 4^+} f(x) = f(4) = 4 \therefore f$ is continuous from the right at $x=4$.
 d. Exercise left for you.

Continuity on interval

- A function is continuous on an interval if and only if it is continuous at every point of the interval.

1. Continuity on open interval

A function f is continuous on an open interval (a, b) if it is continuous at each number inside the interval. i.e., $\lim_{x \rightarrow c} f(x) = f(c)$, for all $c \in (a, b)$.

2. Continuity on closed interval

A function f is continuous on the closed interval $[a, b]$ provided that

- f is continuous on (a, b)
- f is continuous from the right $\lim_{x \rightarrow a^+} f(x) = f(a)$
- f is continuous from the left $\lim_{x \rightarrow b^-} f(x) = f(b)$

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Note: A **continuous function** is one that is continuous at every point of its domain.

The following types are continuous function at every point in their domains.

- Polynomial function are continuous on $(-\infty, \infty)$.
- Absolute value function.
- Exponential function.
- Logarithmic function
- Rational function

Illustrative Examples

50. Find the maximum possible interval on which the function is continuous.

a. $f(x) = x^2 + 6x + 8$ b. $f(x) = \frac{|x-2|}{x-2}$ c. $f(x) = \frac{x^2 - 5x + 6}{x^2 - 4}$

d. $f(x) = \frac{x}{x^2 - 9}$ e. $f(x) = \sqrt{x-4}$ f. $f(x) = \sqrt{9-4x}$

g. $f(x) = \frac{1}{\sqrt{x-3}}$ h. $f(x) = \sqrt{\frac{1-x}{x}}$ i. $f(x) = \sqrt{\frac{4x-3}{x-4}}$

j. $f(x) = \sqrt{\ln \sqrt{x}}$ k. $f(x) = \ln \left(\frac{x-2}{e^x - 1} \right)$ l. $f(x) = x \lfloor x \rfloor$

m. $f(x) = \tan x$

Solution: Here

a. $f(x) = x^2 + 6x + 8$, Domain = $\{x: x \in \mathbb{R}\} \therefore f$ is continuous on $(-\infty, \infty)$

b. $f(x) = \frac{|x-2|}{x-2}$, Domain = $\{x: x \neq 2\} \Rightarrow f$ is discontinuous at $x=2$
 $\therefore f$ is continuous on $(-\infty, 2), (2, \infty)$.

c. $f(x) = \frac{x^2 - 5x + 6}{(x-2)(x+2)}$, Domain = $\{x: x \neq 2 \text{ and } x \neq -2\}$
 $\Rightarrow f$ is continuous on $(-\infty, -2), (-2, 2), (2, \infty)$.

$\therefore f$ is continuous on $(-\infty, -2), (-2, 2), (2, \infty)$

d. $f(x) = \frac{x}{x^2 - 9}$, Domain $\{x: x \neq -3, \text{ and } x \neq 3\}$

$\Rightarrow f$ is discontinuous on at $x = -3$ and $x = 3$.

$\therefore f$ is continuous on $(-\infty, -3), (-3, 3), (3, \infty)$.

e. $f(x) = \sqrt{x-4}$, Domain = $\{x: x-4 \geq 0\} = [4, \infty)$

$\Rightarrow \lim_{x \rightarrow 4^+} f(x) = f(4) = 0$ and $a \in [4, \infty) \Rightarrow \lim_{x \rightarrow a} f(x) = f(a)$

$\therefore f$ is continuous on $[4, \infty)$.

f. $f(x) = \sqrt{9-4x^2}$, Domain = $\left\{x: \frac{-3}{2} \leq x \leq \frac{3}{2}\right\}$

$\therefore f$ is continuous on $\left[\frac{-3}{2}, \frac{3}{2}\right]$

g. $f(x) = \frac{1}{\sqrt{x-3}}$, Domain = $\{x: x > 3\} = (3, \infty)$

h. $f(x) = \sqrt{\frac{1-x}{x}}$, Domain = $\left\{x: \frac{1-x}{x} \geq 0\right\} = \{x: 0 < x \leq 1\}$

$\therefore f$ is continuous on $(0, 1]$.

i. $f(x) = \sqrt{\frac{4x-3}{x-4}}$, Domain = $\left\{x: \frac{4x-3}{x-4} \geq 0\right\} = \left\{x: x \leq \frac{3}{4} \text{ or } x > 4\right\}$

$\therefore f$ is continuous on $\left(-\infty, \frac{3}{4}\right) \cup (4, \infty)$

j. $f(x) = \sqrt{\ln \sqrt{x}}$, Domain = $\{x: x > 0\}$ and $\ln \sqrt{x} \geq 0 \Rightarrow \{x: x \geq 1\}$

$\therefore f$ is continuous on $[1, \infty)$.

k. $f(x) = \ln\left(\frac{x-2}{e^x-1}\right)$, Domain = $\left\{x: \frac{x-2}{e^x-1} > 0\right\}$

Using sign chart, critical point $x = 2$, and $x = 0$.

	$-\infty$	0	2	∞
$x-2$	-	-	+	
e^x-1	-	+	+	
$\frac{x-2}{e^x-1}$	+	-	+	

$\therefore f$ is continuous on $(-\infty, 0) \cup (2, \infty)$.

l. $f(x) = x - [x]$, Domain = $(-\infty, \infty)$

Even if the domain is $(-\infty, \infty)$ but it is not continuous on $\mathbb{R}$.

$\therefore f$ is continuous on $\dots [-2, -1), [-1, 0), [0, 1), [1, 2), \dots$
 $[k, k+1)$, k is integer since $\lim_{x \rightarrow k^-} f(x) = k^- - [k^+] = 0 = f(k)$.

Right side continuity. The greatest integer function has left side discontinuity on $x = k$ where k is an integer.

m $f(x) = \tan x$, Domain $= \left\{ x : x \neq \frac{\pi}{2}k, k \text{ is odd integer} \right\}$

$\therefore f$ is continuous on: $\dots \left(\frac{-3\pi}{2}, \frac{-\pi}{2} \right), \left(\frac{-\pi}{2}, \frac{\pi}{2} \right), \left(\frac{\pi}{2}, \frac{3\pi}{2} \right)$

Finding continuity interval on piece wise defined function

Illustrative Examples

51. Find the possible interval on which the piece wise defined function is continuous.

a $f(x) = \begin{cases} \frac{x^2 + 6x + 8}{x + 2}, & \text{if } x \neq -2 \\ 2 & \text{if } x = -2 \end{cases}$

d. $f(x) = \begin{cases} \frac{x - x^2}{x - 1}, & x < 0 \\ 3 - x, & x \geq 0 \end{cases}$

b. $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ 9, & \text{if } x = 3 \end{cases}$

e. $f(x) = \begin{cases} 4 - x^2, & \text{if } x < 4 \\ x + 1, & \text{if } 1 \leq x < 5 \\ 1 - x, & \text{if } x \geq 5 \end{cases}$

c. $f(x) = \begin{cases} \frac{1}{x - 4} & \text{if } x < 4 \\ x - 4 & \text{if } x \geq 4 \end{cases}$

f. $f(x) = \begin{cases} \frac{1}{x}, & \text{if } -4 \leq x < 4 \\ 1 & \text{if } x = 0 \end{cases}$

g. $f(x) = \begin{cases} \frac{2|x^2 - 4|}{x - 2} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$

h. $f(x) = \begin{cases} \frac{2(x^3 + 8)}{x + 2}, & x \neq -2 \\ 6 & \text{if } x = -2 \end{cases}$

Solution: Here

a. $\lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} \frac{(x+2)(x+4)}{x+2} = 2$ and $f(-2) = 2$

$\therefore f$ is continuous at each $x \in (-\infty, -2)$ and $(-2, \infty)$.

b. $\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} = 6$ and $f(3) = 9$

$$\Rightarrow \lim_{x \rightarrow 3} f(x) \neq f(3) \Rightarrow f \text{ is discontinuous at } x = 3.$$

$\therefore f$ is continuous on $(-\infty, 3)$ and $(3, \infty)$.

c. **To show continuity at $x = 4$.**

$$\lim_{x \rightarrow 4^-} f(x) = \frac{1}{4^- - 4} = \frac{1}{0^-} = \infty \text{ and } f(4) = 0$$

$$\Rightarrow \lim_{x \rightarrow 4^-} f(x) \neq f(4) \text{ thus } f \text{ is discount. at } x = 4.$$

$\therefore f$ is continuous on $(-\infty, 4)$ and $(4, \infty)$

d. **To show continuity at $x = 0$.**

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0^-} \frac{x(1-x)}{x-1} = 0 \text{ and } f(0) = 3$$

$$\Rightarrow \lim_{x \rightarrow 0} f(x) \neq f(0), \text{ thus } f \text{ is discount. at } x = 0$$

$\Rightarrow$ continuous on $(-\infty, 0)$ also, $\lim_{x \rightarrow 0^+} f(x) = f(0)$ thus f is continuous $[0, \infty)$

$\therefore f$ is continuous on $(-\infty, 0)$ and $[0, \infty)$.

e. f is continuous on $(-\infty, 1)$ and $[1, \infty)$. Since it is continuous at $x = 5$.

f. f is continuous on $[-4, 0)$ and $(0, 4)$. Since it is discontinuous at $x = 0$, $\lim_{x \rightarrow 0} f(x) = \infty$ (limit is doesn't exist).

g. **To show continuity at $x = 2$.**

$$\Rightarrow \lim_{x \rightarrow 2^-} f(x) \Leftrightarrow \lim_{x \rightarrow 2^+} \frac{2(x-2)(x+2)}{x-2} = 2(4) = 8 \text{ and}$$

$$\lim_{x \rightarrow 2^-} \frac{2(-(x-2)(x+2))}{x-2} = 2(-1)(4) = -8$$

$$\Rightarrow \lim_{x \rightarrow 2} f(x) \text{ does not exist, so that } f \text{ is discontinuous at } x = 2.$$

$\therefore f$ is continuous on $(-\infty, 2)$ and $(2, \infty)$.

h. **To show continuity at $x = -2$.**

$$\Rightarrow \lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} \frac{2(x+2)(x^2 - 2x + 4)}{x+2} = 4 + 4 + 4 = 12$$

and $f(-2) = 16 \Rightarrow \lim_{x \rightarrow -2} f(x) \neq f(-2)$ so that f is discontinuous at $x = -2$.

$\Rightarrow f$ is continuous on $(-\infty, -2)$ and $(-2, \infty)$.

Determining the value of constant k , from the continuous piece wise defined function

Illustrative Examples

52. Find a value of the constant a , b and k so that each of the given function is continuous on the entire real line.

$$a. \quad f(x) = \begin{cases} x^3, & x \leq 2 \\ ax^2, & x > 2 \end{cases} \quad b. \quad f(x) = \begin{cases} \frac{3}{4}x + \frac{9}{4} & \text{if } x \leq 1 \\ ax + b & \text{if } 1 < x \leq 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$$

$$c. \quad f(x) = \begin{cases} \frac{x^2 - a^2}{x - a} & \text{if } x \neq a \\ 8 & \text{if } x = a \end{cases} \quad d. \quad f(x) = \begin{cases} \frac{\sin 3x}{4x}, & x > 0 \\ k^2 - 2x, & x \leq 0 \end{cases}$$

$$e. \quad f(x) = \begin{cases} 3x + 4, & x \leq 2 \\ kx^2 - 5, & x > 2 \end{cases} \quad f. \quad f(x) = \begin{cases} ax^2 + bx + 4, & \text{if } -2 \leq x < 1 \\ ax + b & \text{if } x < -2 \\ bx + 6 & \text{if } x > 1 \end{cases}$$

$$g. \quad f(x) = \begin{cases} \sqrt{-x^2 + 4x - 3}, & \text{if } 1 \leq x \leq 3 \\ \sqrt{x^2 - 4x + k}, & \text{if } x < 1 \end{cases} \quad h. \quad f(x) = \begin{cases} 6^{|k^2 - 2x + 1|} & \text{if } x > 2 \\ 3x & \text{if } x \leq 2 \end{cases}$$

$$i. \quad \begin{cases} x^2 + 4 & \text{if } x < a \\ 6x - 4 & \text{if } a \leq x \leq b \\ 2x + 8 & \text{if } x > b \end{cases} \quad j. \quad f(x) = \begin{cases} 5x \cot 4x, & x \neq 0 \\ k^2 + 1, & \text{if } x = 0 \end{cases}$$

$$k. \quad \begin{cases} x + 1, & 1 < x < 3 \\ x^2 + kx + c, & |x - 2| \geq 1 \end{cases} \quad l. \quad f(x) = \begin{cases} \frac{kx}{\tan x}, & x \geq 0 \\ k^2 - 2, & x > 0 \end{cases}$$

Solution: Here

a. f is continuous at $x = 2$.

$$\Rightarrow \lim_{x \rightarrow 2^-} f(x) = f(2) \Rightarrow \lim_{x \rightarrow 2^+} ax^2 = 2^3 \Rightarrow 4a = 8 \Leftrightarrow a = 2$$

b. Since f is continuous at $x = 1, 2$.

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) \Rightarrow \lim_{x \rightarrow 1^+} ax + b = \lim_{x \rightarrow 1^-} \frac{3}{4}x + \frac{9}{4}$$

$$\Rightarrow a + b = \frac{3}{4} + \frac{9}{4} = \frac{12}{4} = 3 \dots (*) \text{ and}$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) \Rightarrow \lim_{x \rightarrow 2^+} 2x - 3 = \lim_{x \rightarrow 2^-} ax + b \Rightarrow 4 - 3 =$$

$$2a + b \Rightarrow 2a + b = 1 \dots (**)$$

From (*) and (**): $\begin{cases} a + b = 3 \\ 2a + b = 1 \end{cases} \Rightarrow a = -2 \text{ and } b = 5$

c. Since f is continuous at $x = a$.

$$\Rightarrow \lim_{x \rightarrow a} f(x) = f(a) \Leftrightarrow \lim_{x \rightarrow a} \frac{(x-a)(x+a)}{x-a} = f(a)$$

$$\Leftrightarrow \lim_{x \rightarrow a} (x+a) = 8 \Rightarrow 2a = 8, \therefore a = 4$$

d. Since it is continuous at $x = 0$.

$$\Rightarrow \lim_{x \rightarrow 0} f(x) = f(0) \Leftrightarrow \lim_{x \rightarrow 0} \frac{\sin 3x}{4x} = f(0)$$

$$\Leftrightarrow \frac{3}{4} = k^2 - 0 \Rightarrow k^2 = \frac{3}{4} \quad \therefore k = \frac{\pm\sqrt{3}}{2}$$

e. Since it is continuous at $x = 2$.

$$\Rightarrow \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = f(2)$$

$$\Rightarrow \lim_{x \rightarrow 2} kx^2 - 5 = \lim_{x \rightarrow 2} 3x + 4 \Leftrightarrow 4k - 5 = 10 \Rightarrow 4k = 15 \Leftrightarrow k = \frac{15}{4}$$

f. It is cont. at $x = -2 \Rightarrow \lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^-} f(x)$

$$\Leftrightarrow 4a - 2b + 4 = -2a + b, \text{ also cont. at } 1.$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) \Leftrightarrow b + 6 = a + b + 4 \Rightarrow a = 2$$

Now, put $a = 2$ in $4a - 2b + 4 = -2a + b \Rightarrow 8 - 2b + 4 = -4 + b$

$$\Leftrightarrow 16 = 3b \Rightarrow b = \frac{16}{3} \therefore a = 2 \text{ and } b = \frac{16}{3}$$

g. Since it is continuous at $x = 1$.

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$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} f(x) \Leftrightarrow \lim_{x \rightarrow 1^-} \sqrt{-x^2 + 4x - 3} = \lim_{x \rightarrow 1^-} \sqrt{x^2 - 4x + k}$$

$$\Leftrightarrow \sqrt{-1+4-3} = \sqrt{1-4+k} \Rightarrow \sqrt{k-3} = 0 \Rightarrow k = 3$$

Since it is continuous at $x = 2$.

$$\Rightarrow \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} f(x) \Leftrightarrow \lim_{x \rightarrow 2^+} 6^{|k^2 - 2x + 1|} = \lim_{x \rightarrow 2} 3x$$

$$\Leftrightarrow 6^{k^2-1} = 6^1 \Rightarrow |k^2 - 3| = 1 \Rightarrow k^2 - 3 = 1 \text{ or } k^2 - 3 = -1$$

$$\Rightarrow k = 2, -2, -\sqrt{2}, \sqrt{2} \therefore \text{The solution set} = \{-2, 2, -\sqrt{2}, \sqrt{2}\}$$

i. Since it is continuous at $x = a$, and $x = b$

$$\Rightarrow \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} f(x) \Leftrightarrow 6a - 4 = a^2 + 4 \Leftrightarrow a^2 - 6a + 8 = 0$$

$$\Leftrightarrow (a-4)(a-2) = 0 \Rightarrow a = 4 \text{ and } a = 2 \text{ Also,}$$

$$\lim_{x \rightarrow b^+} f(x) = \lim_{x \rightarrow b^+} f(x) \Leftrightarrow 2b + 8 = 6b - 4 \Rightarrow b = 3$$

$\therefore a = 2$ and $b = 3$, reject $a = 4$.

$$j. \lim_{x \rightarrow 0} f(x) = f(0) \Rightarrow \lim_{x \rightarrow 0} 5x \cot 4x = k^2 = 1$$

$$\Leftrightarrow \lim_{x \rightarrow 0} \frac{5x \cos 4x}{\sin 4x} = k^2 + 1 \Leftrightarrow \lim_{x \rightarrow 0} \frac{5x}{\sin(4x)} \lim_{x \rightarrow 0} \cos 4x = k^2 + 1$$

$$\Leftrightarrow \frac{5}{4} = k^2 + 1 \Leftrightarrow k^2 = \frac{5}{4} - 1 = \frac{1}{4} \Rightarrow k = -\frac{1}{2}, \frac{1}{2}$$

$$\therefore k = -\frac{1}{2} \text{ and } k = \frac{1}{2}$$

k. Exercise left for you.

$$l. \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} f(x) \Leftrightarrow \lim_{x \rightarrow 0^+} \frac{kx}{\tan x} = k^2 - 2 \Leftrightarrow k^2 - 2 = k \Rightarrow k^2 - k - 2 = 0 \Rightarrow (k-2)(k+1) = 0 \Leftrightarrow k = 2 \text{ or } k = -1$$

Continuity of Compositions of function

Theorem: If $\lim_{x \rightarrow a} g(x) = L$ and if the function f is continuous at L ,

then $\lim_{x \rightarrow a} f(g(x)) = f(L)$ that is $\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$.

Illustrative Examples

53. Find the maximum possible interval on which f is continuous.

a. $f(x) = \sqrt{25-x}$ b. $f(x) = \sqrt{1-\sqrt{x^2-4}}$

c. $f(x) = g(k(x))$, $g(x) = \sqrt{x}$ and $k(x) = x^2 - 2x$

Solution: Here

a. $f(x) = \sqrt{25-x^2}$, Domain = $\{x: -5 \leq x \leq 5\}$; f is continuous on $[-5, 5]$

b. $f(x) = \sqrt{1-\sqrt{x^2-4}} \Rightarrow \text{Domain} = \{x: 1 - \sqrt{x^2-4} \geq 0 \text{ and } x^2-4 \geq 0\} = \{x: x^2-4 \leq 1 \text{ and } |x| \geq 2\} = \{x: |x| \leq \sqrt{5} \text{ and } |x| \geq 2\}$
 $= \{x: -\sqrt{5} \leq x \leq \sqrt{5} \text{ and } -\infty < x \leq -2 \cup 2 \leq x < \infty\}$
 $= \{x: -\sqrt{5} \leq x \leq -2 \cup 2 \leq x \leq \sqrt{5}\}$

c. $f(x) = g(k(x)) = g(x^2 - 2x) = \sqrt{x^2 - 2x}$
 $\therefore \text{Domain} = \{x: x^2 - 2x \geq 0\} = \{x: x \leq 0 \text{ and } x \geq 2\}$
 $\therefore f$ is continuous on $(-\infty, 0]$ and $[2, \infty)$.

Intermediate value theorem

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

The intermediate value theorem often can be used to locate the zeros of function on $[a, b]$ for which $f(a)$ and $f(b)$ differ in sign.

Illustrative Examples

54. Use the intermediate value theorem to show that f has zero between a and b .

a. $f(x) = 2x^3 + 5x^2 - 3$, $a = -3$, $b = -2$

b. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$, $a = 1$, $b = 2$

c. $f(x) = x^3 + 2x - 1$, $a = 0$ and $b = 1$

Solution: Here

a. $f(-3) = -54 + 45 - 3 = -12 < 0$
 $f(-2) = -16 + 20 - 3 = 1 > 0$

$\therefore c \in [-3, -2]$ such that $f(c) = 0$

- b. $f(1) = 1 + 2 - 6 + 2 - 3 = -4 < 0$ and $f(2) = 32 + 32 - 48 + 4 - 3 = 17 > 0 \Rightarrow f(1) < 0$ and $f(2) > 0 \therefore c \in [1, 2]$ such that $f(c) = 0$
- c. Exercise left for you.

15. Use the bisection method to find (approximate) the square root.

a. $\sqrt{2}$ b. $\sqrt{5}$

Solution. Let $x = \sqrt{2} \Rightarrow x^2 - 2 = 0 \Rightarrow f(x) = x^2 - 2, f(1) = -1 < 0$ and $f(2) = 2 > 0 \Rightarrow f$ has real root in $(1, 2)$ so that using bisection method.

Opposite sign in interval (a, b)	Mid.pt c	Sign of f			
		f(a)	f(c)	f(b)	
(1, 2)	1.5	-	+	+	$\leftarrow (a, c)$
(1, 1.5)	1.25	-	-	+	$\leftarrow (c, b)$
(1.25, 1.5)	1.375	-	-	+	$\leftarrow (c, b)$
(1.375, 1.5)	1.4375	-	+	+	$\leftarrow (a, c)$
(1.375, 1.4375)	1.40625	-	-	+	$\leftarrow (c, b)$
(1.40625, 1.4375)	1.421875	-	+	+	$\leftarrow (a, c)$

$\therefore \sqrt{2} \approx 1.41$

b. Exercise left for you.

Continuously compounded interest

The compound interest formula is $A = P \left(1 + \frac{r}{n} \right)^{nt}$ let

$\frac{1}{k} = \frac{r}{n} \Rightarrow n = kr$ and $nt = krt$, so that we can transform as

$$A = P \left(1 + \frac{1}{k} \right)^{krt} = P \left[\left(1 + \frac{1}{k} \right)^k \right]^{rt}, \text{ as } n \rightarrow \infty, k \rightarrow \infty \text{ using the fact that}$$

$$\left(1 + \frac{1}{k} \right)^k \rightarrow e \text{ as } k \rightarrow \infty \therefore P \left[\left(1 + \frac{1}{k} \right)^k \right]^{rt} = P(e)^{rt}, \text{ where}$$

- P = principal
- r = annual interest rate expressed as decimal
- t = number of years P is invested
- A = amount after t year
- n = number of interest period per year

Illustrative Example

56. Suppose birr 20,000 is deposited in an account paying at rate of 8% annual interest compounded continuously. Determine the balance in the account:

a. after 5 year b. after 10 year and 3 months

Solution: a. $P = 20,000$, $r = 0.08$ and $t = 5$ year $\Rightarrow A = Pe^{rt} = 20,000 e^{0.08(5)} = 20,000 e^{0.4}$

b. $t = 10$ year and 3 months $= 10 + \frac{3}{12} = 10 + 0.25 = 10.25 \Rightarrow A =$

$$Pe^{rt} = 20,000 e^{0.08(10.25)} = 20,000 e^{0.82}$$

57. An investment of Birr 10,000 increased to Birr 20,000 in 5 years. If interest was compounded continuously. Find the interest rate.

Solution: $A = Pe^{rt} \Rightarrow 20,000 = 10,000 e^{5r} \Leftrightarrow e^{5r} = 2 \Rightarrow r = \frac{\ln 2}{5}$

L' Hopital's Rule

L' Hopital's rule is used to calculate the limit of the function which

determinant form $\frac{0}{0}$.

Theorem: Suppose that $f(a) = 0$, $g(a) = 0$ that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

Note: L' hopital's rule applied to limits which gives to only $\frac{0}{0}$ form.

Illustrative Example

58. Use L' hopital's rule to find the limit.

a. $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2}$

b. $\lim_{x \rightarrow 1} \frac{x^{10} - 1}{x^6 - 1}$

c. $\lim_{t \rightarrow 8} \frac{x^t - t^x}{x - t}$

d. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin x}$

e. $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$

f. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{x^2 - \frac{\pi^2}{4}}$

g. $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}}$

h. $\lim_{x \rightarrow 1} \frac{\sin \pi x}{x - 1}$

i. $\lim_{x \rightarrow 0} \frac{\sin x}{x^2}$

$$j. \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \quad k. \lim_{x \rightarrow \frac{\pi}{3}} \frac{\cos x - \frac{1}{2}}{x - \frac{\pi}{3}} \quad l. \lim_{x \rightarrow 0} \frac{\sec x}{1 + \tan x}$$

Solution: Here

$$a. \lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2} \rightarrow \frac{0}{0} \text{ form } \Leftrightarrow \lim_{x \rightarrow -2} \frac{2x}{1} = -2, \text{ Not } \frac{0}{0}, \text{ limit is found}$$

$$b. \lim_{x \rightarrow 1} \frac{x^{10} - 1}{x^6 - 1} \Rightarrow \lim_{x \rightarrow 1} \frac{10x^9}{6x^5} = \frac{10}{6} = \frac{5}{3}$$

$$c. \lim_{t \rightarrow x} \frac{x^n - t^n}{x - t} = \lim_{t \rightarrow x} nt^{n-1} = nx^{n-1}$$

$$d. \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin x} = \lim_{x \rightarrow 0} \frac{e^{2x}(2)}{\cos x} = \frac{2}{1} = 2$$

$$e. \lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi} = \lim_{x \rightarrow \pi} \frac{\cos x}{1} = \cos \pi = -1$$

$$f. \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{x^2 - \frac{\pi}{4}} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\sin x}{2x} = \frac{-\sin\left(\frac{\pi}{2}\right)}{2\left(\frac{\pi}{2}\right)} = -\frac{1}{\pi}$$

$$g. \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sec^2 x}{1} = \frac{2}{4} = \frac{1}{2}$$

$$h. \lim_{x \rightarrow 1} \frac{\sin \pi x}{x - 1} = \lim_{x \rightarrow 1} (\cos \pi x)(\pi) = -\pi$$

$$i. \lim_{x \rightarrow 0} \frac{\sin x}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x}{2x}, \text{ Not } \frac{0}{0} \text{ limit is found } \Rightarrow \lim_{x \rightarrow 0^+} \frac{\cos x}{2x} = \infty$$

$$\text{and } \lim_{x \rightarrow 0^-} \frac{\cos x}{2x} = -\infty$$

$$\therefore \lim_{x \rightarrow 0^+} \frac{\sin x}{x^2} = \infty \text{ and } \lim_{x \rightarrow 0^-} \frac{\sin x}{x^2} = -\infty$$

$$j. \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \leftarrow \text{still } \frac{0}{0}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin x}{6x} \leftarrow \text{still } \frac{0}{0} \Rightarrow \lim_{x \rightarrow 0} \frac{\cos x}{6} = \frac{\cos(0)}{6} = \frac{1}{6}$$

$$\therefore \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \frac{1}{6} \leftarrow \text{wonder! L' hospital (Bravo)!}$$

$$k. \text{ Exercise left for you. } \left(\text{Ans. } \frac{-\sqrt{3}}{2} \right)$$

l. Exercise left for you. (Ans. 1)

Solved Problem

59. Which of the following is true about $\left\{ 2 + \frac{(-1)^n}{n+1} \right\}_{n=1}^{\infty}$... (UEE)

A. The least upper bound of sequence is $\frac{7}{3}$.

B. The sequence is monotonic.

C. 1.6 is its lower bound.

D. $2 - \frac{1}{15}$ is term of the sequence.

Solution: For odd number: $n = 1, 3, 5, \dots, \infty$.

$$\Rightarrow \left\{ 2 + \frac{-1}{n+1} \right\} = \left\{ \frac{2n+1}{n+1} \right\}_{n=1}^{\infty} = \left\{ \frac{3}{2}, \frac{7}{4}, \dots, 2 \right\} \leftarrow \text{increasing.}$$

For even number: $n = 2, 4, 6, \dots, \infty$.

$$\begin{aligned} \Rightarrow \left\{ 2 + \frac{1}{n+1} \right\} &= \left\{ \frac{2n+3}{n+1} \right\}_{n=2}^{\infty} \\ &= \left\{ \frac{7}{3}, \frac{11}{5}, \dots, 2 \right\} \leftarrow \text{decreasing} \Rightarrow \frac{3}{2} \leq \left\{ 2 + \frac{(-1)^n}{n+1} \right\}_{n=1}^{\infty} \leq \frac{7}{3} \end{aligned}$$

$$\Rightarrow \text{glb} = \frac{3}{2} \leftarrow \text{greatest lower bound of the sequence.}$$

$\Rightarrow \text{lub} = \frac{7}{3} \leftarrow \text{least upper bound of sequence.}$

$\Rightarrow$ The sequence is not monotonic (varies).

$\Rightarrow$ When $n = 14$, $2 + \frac{1}{14+1} = \frac{31}{15}$, thus $2 - \frac{1}{5}$ is not term of sequence.

Answer: A

Which of the following is not true about the sequence $\left\{ \frac{6^n}{n!} \right\}_{n=1}^{\infty}$?

a. The sequence is increasing.

b. The glb of sequence is 0.

c. The lub of sequence is $\frac{6^5}{5!}$.

d. The sequence is convergent.

Solution: To decide the monotonicity let us use ratio test $\frac{a_{n+1}}{a_n}$ let

$$a_n = \frac{6^n}{n!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{6^{n+1}}{(n+1)!} \cdot \frac{n!}{6^n} = \frac{6^n \cdot 6}{(n+1)n!} \cdot \frac{n!}{6^n} = \frac{6}{n+1} \text{ so that}$$

$\frac{a_{n+1}}{a_n} \geq 1$, when $n = 1, 2, 3, 4, 5 \Rightarrow$ the sequence is increasing

only from $n = 1, 2, 3, 4, 5$. However, $\frac{a_{n+1}}{a_n} < 1$ for all $n \geq 6$.

$\Rightarrow$ The sequence is eventually and strictly decreasing.

$\Rightarrow 0 < \left\{ \frac{6^n}{n!} \right\} \leq \frac{6^5}{5!} \Rightarrow \text{glb } 0 \text{ and lub } = \frac{6^5}{5!}$ since the sequence is

bounded and decreasing. So that the sequence is convergent and converges to glb which is zero.

Answer: A

The greatest lower bound of the sequence $\{-1.4, -1.44, -1.444, \dots\} \dots$ (UEE)

- a. $-\frac{13}{10}$ B. $-\frac{13}{9}$ C. -1.4 D. -1.5

Solution: $\{-1.4, -1.44, -1.444, \dots\} = \{1 + 4(0.1, 0.11, 0.111, \dots)\}$

$$= -\left\{1 + \frac{4}{9}(0.9, 0.99, \dots)\right\} = -\left\{1 + \frac{4}{9}\left(1 - \frac{1}{10}, 1 - \frac{1}{100}, \dots\right)\right\}$$

that the n^{th} term of sequence (generator)

$$\left[1 + \frac{4}{9} + \left\{1 - \frac{1}{10^n}\right\}_{n=1}^{\infty}\right]. \text{ The sequence is decreasing to } 1 + \frac{4}{9}$$

$$\Rightarrow \text{lub} = -1.4 \text{ and glb} = -\left(1 + \frac{4}{9}\right) = -\frac{13}{9} \text{ as } n \rightarrow \infty. \text{ Answer: B}$$

62. The sequence $\left\{\frac{\sqrt{n+1}-3}{1-\sqrt{n}}\right\}_{n=2}^{\infty}$ converges to: (UEE)
- a. $-\infty$ B. -3 C. -1 D. 1

$$\text{Solution: } \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}-3}{1-\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{1}{n}} - 3}{\frac{1}{\sqrt{n}} - 1} = \frac{1-3}{0-1} = -2$$

Answer: C

63. Let $a_n = \frac{1-\sqrt{n}}{3n+5}$, then $\lim_{n \rightarrow \infty} a_n =$ _____
- a. $-\frac{1}{3}$ B. 0 C. $\frac{1}{3}$ D. $-\infty$

$$\text{Solution: } \lim_{n \rightarrow \infty} \frac{1-\sqrt{n}}{3n+5} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} - \frac{\sqrt{n}}{n}}{3 + \frac{5}{n}} = \frac{0-0}{3+0} = 0 \text{ Answer: B}$$

64. $\lim_{n \rightarrow \infty} \frac{3\sqrt{n}-6n+5}{4n+1}$ (UEE)

a. $-\frac{3}{2}$ B. 0 C. $\frac{3}{4}$ D. $\frac{5}{4}$

Solution: $\lim_{n \rightarrow \infty} \frac{3\sqrt{n} - 6n + 5}{4n + 1} = \lim_{n \rightarrow \infty} \frac{3\sqrt{\frac{1}{n}} - 6 + \frac{5}{n}}{4 + \frac{1}{n}} = \frac{0 - 6 + 0}{4 + 0} = -\frac{3}{2}$

6. $\lim_{n \rightarrow \infty} \left\{ \frac{1 + 4 + 7 + \dots + 3n - 2}{5 - 2n^2} \right\}$ is equal to _____

a. $-\frac{3}{2}$ B. $\frac{3}{10}$ C. $-\frac{3}{4}$ D. $\frac{3}{4}$

Solution: Since it is standard arithmetic progression so that nth

partial sum: $S_n = \frac{n}{2} [A_1 + A_n] \Leftrightarrow S_n = \frac{n}{2} [1 + 3n - 2] = \frac{3n^2 - n}{2}$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{3k - 2}{5 - 2n^2} \right) = \lim_{n \rightarrow \infty} \frac{3n^2 - n}{10 - 4n^2} = \lim_{n \rightarrow \infty} \frac{3 - \frac{1}{n}}{\frac{10}{n^2} - 4} = -\frac{3}{4}$$

Answer: C

7. $\lim_{x \rightarrow 2^+} \frac{5x|x-2|}{2-x}$ is equal to:

a. -10 B. 10 C. 0 D. doesn't exist

Solution: $|x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$ so that

$$\lim_{x \rightarrow 2^+} \frac{5x|x-2|}{2-x} = \lim_{x \rightarrow 2^+} \frac{5x(x-2)}{2-x} = \lim_{x \rightarrow 2^+} 5(x)(-1) = -10$$

Answer: A

8. If $f(x) = \sqrt{3-x} - 1$ then $\lim_{x \rightarrow 2} \frac{f(x)}{x-2}$

a. 1 B. $-\frac{1}{2}$ C. ∞ D. $-\infty$

Solution: $\lim_{x \rightarrow 2} \frac{f(x)}{x-2} = \lim_{x \rightarrow 2} \frac{\sqrt{3-x}-1}{x-2} \Rightarrow \lim_{x \rightarrow 2} \frac{\sqrt{3-x}-1}{x-2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}$

$$= \lim_{x \rightarrow 2} \frac{3-x-1}{(x-2)(\sqrt{3-x}+1)}$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{2-x}{(x-2)\sqrt{3-x}+1} = \lim_{x \rightarrow 2} \frac{-1}{\sqrt{3-x}+1} = \frac{-1}{1+1} = -\frac{1}{2}$$

Answer: B

68. $\lim_{x \rightarrow \infty} \left[\frac{x}{1+x} \right]^{x+10}$ is equal to (UEE)

a. e^{10} B. e^{-1} C. e^9 D. e

Solution:

$$\lim_{x \rightarrow \infty} \left[\frac{x}{1+x} \right]^{x+10} \Rightarrow \lim_{x \rightarrow \infty} \left[\frac{1}{1+\frac{1}{x}} \right]^x \cdot \lim_{x \rightarrow \infty} \left[\frac{1}{1+\frac{1}{x}} \right]^{10} \Rightarrow \frac{1}{e} \cdot 1 = \frac{1}{e}$$

Answer: B

69. If $f(x) = \begin{cases} \frac{\sin(2a+1)x}{3x}, & x < 0 \\ \frac{\tan(8b-1)x}{5x}, & x > 0 \end{cases}$ and

if $\lim_{x \rightarrow 0} f(x) = 3$, find the value of a and b .

Solution:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin(2a+1)x}{3x} = \frac{2a+1}{3} = 3 \Rightarrow a = 4$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\tan(8b-1)x}{5x} = \frac{8b-1}{5} = 3 \Rightarrow b = 2$$

70. $\lim_{x \rightarrow \infty} \left[\frac{x}{x-2} \right]^{x+1}$ is equal to _____

a. e^{-2} B. e^2 C. e^4 D. 1

70. **Solution:** $\lim_{x \rightarrow \infty} \left[\frac{1}{1 - \frac{2}{x}} \right]^x = \lim_{x \rightarrow \infty} \frac{1}{\left(1 - \frac{2}{x}\right)^x} = \frac{1}{e^{-2}} = e^2$, Answer: B

71. $\lim_{x \rightarrow \infty} \left[\frac{x+4}{x-1} \right]^x$ is equal to _____ (UEE 2000)
 a. e^5 B. 0 C. e^4 D. e^3

Solution: $\lim_{x \rightarrow \infty} \left[\frac{x+4}{x-1} \right]^x = \frac{\lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^x}{\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x} = \frac{e^4}{e^{-1}} = e^5$, Answer: C

72. $\lim_{x \rightarrow 1} \frac{x^3 + x - 2}{x^2 - x}$ is equal to ... (UEE, 2002)
 a. -2 B. 1 C. 4 D. doesn't exist

Solution:

$$\lim_{x \rightarrow 1} \frac{(x-1)(x^2 + x + 2)}{(x-1)(x)} = \lim_{x \rightarrow 1} \frac{x^2 + x + 2}{x} = 1 + 1 + 2 = 4$$

73. $\lim_{x \rightarrow 0^-} \frac{|x| - 4xe^x}{x}$ is equal to:
 a. 0 B. -4 C. -3 D. -5

Solution: $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$, so that

$$\Rightarrow \lim_{x \rightarrow 0^-} \frac{-x - 4xe^x}{x} = \lim_{x \rightarrow 0^-} \frac{-x(1 + 4e^x)}{x} = (-1)(1 + 4) = -5$$

74. $\lim_{x \rightarrow \pi} \frac{x \sin 2x}{x - \pi}$ is equal to ... (UEE)
 a. 2π B. π C. 0 D. 1

Solution: Let $y = x - \pi \Rightarrow x = y + \pi, y \rightarrow 0$

$$\Rightarrow \lim_{y \rightarrow 0} (y + \pi) \frac{\sin(2y + 2\pi)}{y} = \pi \lim_{y \rightarrow 0} \frac{\sin 2y}{y} = 2\pi \quad \text{Answer: A}$$

75. $\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2}$ is equal to _____

a. $\frac{a+b}{2}$ B. $\frac{a-b}{2}$ C. $\frac{b^2-a^2}{2}$ D. $\frac{a^2-b^2}{2}$

Answer: C

76. $\lim_{x \rightarrow 0} \frac{6x - \sin 4x}{2x + 3 \sin 2x}$ is equal to _____

A. $\frac{1}{2}$ B. $\frac{1}{4}$ C. $\frac{3}{8}$ D. $\frac{3}{4}$

Solution: $\lim_{x \rightarrow 0} \frac{6x - \sin 4x}{2x + 3 \sin 2x} = \lim_{x \rightarrow 0} \frac{6x - \sin 4x}{2x + 3 \sin 2x} \cdot \frac{x}{x}$

$$\Rightarrow \frac{\lim_{x \rightarrow 0} \left(6 - \frac{\sin 4x}{x} \right)}{\lim_{x \rightarrow 0} \left(2 + 3 \frac{\sin 2x}{x} \right)} = \frac{6 - 4}{2 + 3(2)} = \frac{2}{8} = \frac{1}{4}, \quad \text{Answer: B}$$

77. If $f(x) = \frac{1}{\sqrt{4-x^2}}$, then $\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x-1}$,

A. $\frac{1}{3\sqrt{3}}$ B. $\frac{-1}{\sqrt{3}}$ C. $\frac{1}{2\sqrt{6}}$ D. $\frac{1}{6\sqrt{3}}$

Solution: $\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{1}{\sqrt{4-x^2}} - \frac{1}{\sqrt{3}}}{x-1}$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\sqrt{3} - \sqrt{4-x^2}}{\sqrt{3}\sqrt{4-x^2} \cdot (x+1)} \cdot \frac{\sqrt{3} + \sqrt{4-x^2}}{\sqrt{3} + \sqrt{4-x^2}}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{3 - (4 - x^2)}{(x-1)(\sqrt{3})\sqrt{4-x^2} \cdot (\sqrt{3} + \sqrt{4-x^2})}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)(\sqrt{3})\sqrt{4-x^2}(\sqrt{3} + \sqrt{4-x^2})} \leftarrow \text{cancelling } \frac{x}{x}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{x+1}{(\sqrt{3})(\sqrt{4-x^2})(\sqrt{3} + \sqrt{4-x^2})} = \frac{2}{\sqrt{3} \cdot \sqrt{3} (\sqrt{3} + \sqrt{3})}$$

$$= \frac{2}{6\sqrt{3}} = \frac{1}{3\sqrt{3}}$$

∴ Answer: ...

$$\lim_{x \rightarrow 2} \frac{\sqrt{x-2} + \sqrt{x} - \sqrt{2}}{\sqrt{x^2-4}} \text{ is equal to:}$$

A. $\frac{1}{4}$

B.

$\frac{1}{2}$

C.

$\frac{1}{\sqrt{2}}$

D.

Solution: Note $\sqrt{x^2-4} = \sqrt{(x-2)(x+2)} = \sqrt{x-2} \cdot \sqrt{x+2}$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{1}{\sqrt{x+2}} \left(\frac{\sqrt{x-2}}{\sqrt{x-2}} + \frac{\sqrt{x} - \sqrt{2}}{\sqrt{x-2}} \cdot \frac{(\sqrt{x} + \sqrt{2})}{(\sqrt{x} + \sqrt{2})} \right)$$

$$\Rightarrow \frac{1}{\sqrt{4}} \lim_{x \rightarrow 2} \left(1 + \frac{x-2}{\sqrt{x-2}} \cdot \frac{1}{\sqrt{x+2}} \right) = \frac{1}{2} \left(1 + \lim_{x \rightarrow 2} (x-2) \right) \cdot \frac{1}{2}$$

$$= \frac{1}{2} + 0 = \frac{1}{2}$$

Answer: B

$$\lim_{x \rightarrow 0} \left(\frac{x^2-3}{x} \right) \sin x \text{ is equal to:}$$

A. 0

B. 1

C. -3

D. doesn't exist

Solution: $\Rightarrow \lim_{x \rightarrow 0} (x^2-3) \cdot \lim_{x \rightarrow 0} \frac{\sin x}{x} = (-3)(1) = -3$

Answer: C

80. If $f(x) = \begin{cases} \frac{x^2 - 4}{x + 2}, & \text{if } x \neq -2 \\ kx + 5, & \text{if } x = -2 \end{cases}$ then the value of k , so that f is continuous at $x = -2$.

A. $\frac{9}{2}$ B. $-\frac{9}{2}$ C. -4 D. 9

Solution: Since it is continuous at $x = -2 \Rightarrow \lim_{x \rightarrow -2} f(x) = f(-2)$

$$\Rightarrow \lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2} = -2k + 5 \Rightarrow \lim_{x \rightarrow -2} \frac{(x - 2)(x + 2)}{x + 2}$$

$$= -2k + 5 \Rightarrow -4 = -2k + 5 \Rightarrow -2k = -9 \Rightarrow k = \frac{9}{2}$$

Answer: A

81. Let $f(x) = \begin{cases} a - \cos x, & x \leq 0 \\ 3e^x - a, & x > 0 \end{cases}$, for what value of 'a' is continuous function. UEE 2002 E.C

A. 1 B. $\frac{3}{2}$ C. 2 D. 3

Solution: Let f cont. at $x = 0$.

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) \Rightarrow \lim_{x \rightarrow 0^+} 3e^x - a = \lim_{x \rightarrow 0^-} (a - \cos x)$$

$$\Rightarrow 3(1) - a = a - 1 \Rightarrow 4 = 2a \Rightarrow a = 2$$

Answer: C

82. Let $f(x) = \begin{cases} 3a - x, & \text{if } x < 1 \\ b + 1 & \text{if } x = 1 \\ ax & \text{if } x > 1 \end{cases}$, what is the value of b if f is continuous at $x = 1$.

A. $-\frac{1}{2}$ B. $\frac{1}{2}$ C. -1 D. 1

Solution: $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) \Rightarrow \lim_{x \rightarrow 1^+} ax = \lim_{x \rightarrow 1^-} (3a - x)$

$$\Rightarrow a = 3a - 1 \Rightarrow -2a = -1 \Rightarrow a = \frac{1}{2}, \text{ and}$$

$$\lim_{x \rightarrow 1^-} f(x) = f(1) \Rightarrow \lim_{x \rightarrow 1^-} ax = b + 1 \Rightarrow a = b + 1 \text{ so that}$$

$$\frac{1}{2} = b + 1 \Rightarrow b = \frac{1}{2} - 1 = -\frac{1}{2}$$

Answer: A

On which of the following intervals is f continuous, where

$$f(x) = \begin{cases} \frac{x-x^2}{x-1}, & x < 0 \\ 3-x, & x \geq 0 \end{cases}$$

A. $(-\infty, 0]$ B. $[-2, 1]$ C. $[-1, 2]$ D. $[0, \infty)$

Solution: To show f is cont. on $(-\infty, 0]$.

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = f(0) \Rightarrow \lim_{x \rightarrow 0^-} \frac{x(1-x)}{x-1} = 3 - 0 \Rightarrow 0 \neq 3 \therefore f \text{ is}$$

not cont. from the left 0.

$$\text{Let } t \in (-\infty, 0) \quad \lim_{x \rightarrow t} f(x) = f(t) \Rightarrow \lim_{x \rightarrow t} \frac{x(1-x)}{x-1} = \frac{t(1-t)}{t-1},$$

$\Rightarrow f$ is cont. on $(-\infty, 0)$ but f is not cont. on $(-\infty, 0]$

d. To show f is cont. on $[0, \infty)$, let $t \in [0, \infty)$

$$\Rightarrow \lim_{x \rightarrow t} f(x) = f(t) \Rightarrow \lim_{x \rightarrow t} 3 - x = 3 - t \Rightarrow f \text{ is cont. on}$$

$[0, \infty) \therefore \text{Answer: D}$

$$84. \text{ Let } f(x) = \begin{cases} \frac{a}{x^2 + 3} & \text{if } x \leq 1 \\ \frac{x-1}{\sqrt{x}-1} & \text{if } x > 1 \end{cases}$$

, what is the value of a if f is

continuous at $x = 1$.

A. 0

B. 2

C. 4

D. 8

$$\text{Solution: } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) \Rightarrow \lim_{x \rightarrow 1^-} \frac{a}{x^2 + 3} = \lim_{x \rightarrow 1^+} \frac{x-1}{\sqrt{x}-1}$$

$$\Rightarrow \frac{a}{4} = \lim_{x \rightarrow 1^+} \frac{x-1}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x \rightarrow 1^+} \frac{x-1}{x-1} (\sqrt{x}, 1) = 2$$

$$\frac{a}{4} = 2 \Rightarrow a = 8,$$

Answer: D

Supplementary Exercise on Unit - 11

1. Find i) the set of all upper bound.
 ii) the smallest of all upper bound (lub)
 iii) the set of all lower bound
 iv) the largest of all lower bound of glb

a. $\left\{(-1)^n + \frac{1}{n+2}\right\}_{n=1}^{\infty}$ b. $\left\{\frac{n-2}{n+3}\right\}_{n=1}^{\infty}$ c. $\left\{\frac{n+3}{4n-5}\right\}_{n=1}^{\infty}$

d. $\{ne^{-2n}\}_{n=1}^{\infty}$ e. $\left\{\frac{4^n}{n!}\right\}_{n=1}^{\infty}$ f. $\left\{\frac{(-1)^n + 1}{3}\right\}_{n=1}^{\infty}$

2. Determine the given sequence $\{a_n\}$ is strictly increasing or strictly decreasing.

a. $\left\{4 - \frac{1}{n}\right\}_{n=1}^{\infty}$ b. $\left\{4 + \frac{1}{n+1}\right\}_{n=1}^{\infty}$ c. $\left\{\frac{n}{3n+1}\right\}_{n=1}^{\infty}$

d. $\left\{\frac{2-3n}{2n+1}\right\}_{n=1}^{\infty}$ e. $\left\{\sin\left(\frac{\pi}{3n}\right)\right\}_{n=1}^{\infty}$ f. $\left\{\cos\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$

g. $\left\{(-1)^n + \frac{1}{n+1}\right\}_{n=1}^{\infty}$ h. $\{(-1)^n + 1\}_{n=1}^{\infty}$ i. $\left\{\frac{2n-1}{2-3n}\right\}_{n=1}^{\infty}$

j. $\left\{\frac{n^n}{n!}\right\}_{n=1}^{\infty}$ k. $\left\{\frac{n!}{3^n}\right\}_{n=1}^{\infty}$

3. Determine whether the sequence converges and if so find its limit.

a. $\left\{\frac{n}{n+2}\right\}_{n=1}^{\infty}$ b. $\left\{(-1)^n \frac{2n^3}{n^3+1}\right\}_{n=1}^{\infty}$ c. $\left\{\frac{(n-1)(n+2)}{2n^2}\right\}_{n=1}^{\infty}$

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d. $\{2\}_{n=1}^{\infty}$ e. $\left\{\left(\frac{n+3}{n+1}\right)^n\right\}_{n=1}^{\infty}$ f. $\left\{\frac{n^n}{n!}\right\}_{n=1}^{\infty}$

Find the limit of the following sequence.

a. $\{2, 7, 2.77, 2.777, \dots\}$ b. $\{-1.2, -1.22, -1.222, \dots\}$

Find the limit value:

a. $\lim_{n \rightarrow \infty} \frac{4n^3 + 5n - 1}{6n^3 + 7n + 4}$

c. $\lim_{n \rightarrow \infty} \frac{\sqrt{3n^2 + 6}}{4 - 2n}$

b. $\lim_{n \rightarrow \infty} \frac{6 - 5n^2}{4n + 15n^2}$

d. $\lim_{n \rightarrow \infty} \frac{2\sqrt{n} - 5n}{15n - 1}$

Find the limit:

a. $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \dots + \frac{n}{n^2} \right)$ b. $\lim_{n \rightarrow \infty} \frac{(2 + 5 + 8 + \dots + 3n - 1)}{4 - 5n^2}$

Find the limit:

a. $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x^2 - 3x - 10}$

b. $\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 + x - 2}$

c. $\lim_{x \rightarrow 6} \frac{\frac{1}{x-1} - \frac{1}{x-6}}{\frac{1}{x-6}}$

d. $\lim_{x \rightarrow a} \frac{\frac{1}{x^3} - \frac{1}{a^3}}{\frac{1}{x} - \frac{1}{a}}$

e. $\lim_{t \rightarrow x} \frac{x^n - t^n}{x - t}$

f. $\lim_{x \rightarrow 0} \frac{\frac{1}{3-x} - \frac{1}{3}}{x}$

g. $\lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x - 4}$

h. $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4}$

i. $\lim_{x \rightarrow 2} \frac{(x+3)^{\frac{1}{3}} - \sqrt[3]{5}}{x - 2}$

j. $\lim_{x \rightarrow 1} \frac{(x+2)^2 - 9}{x - 1}$

k. $\lim_{x \rightarrow 3} \frac{x - 3}{\sqrt{x - 2} - \sqrt{4 - x}}$

l. $\lim_{x \rightarrow 3} \frac{\sqrt{x-3} + \sqrt{x} - \sqrt{3}}{\sqrt{x^2 - 9}}$

Find the limit:

a. $\lim_{x \rightarrow \infty} \left[\frac{3 + 4x}{5 + 4x} \right]^{2x}$

c. $\lim_{x \rightarrow 2} \left(\frac{x+3}{x-2} \right)^{x+5}$

1.1

$$b. \lim_{x \rightarrow 2} \left(\frac{3x}{3x-4} \right)^{5x}$$

$$d. \lim_{x \rightarrow 2} \left(\frac{2x-2}{2x} \right)^{4x}$$

9. Find the value of k :

$$a. \lim_{x \rightarrow 2} \left(\frac{x-k}{x+k} \right)^x = 5$$

$$b. \left(\frac{2x-k}{2x} \right)^{3x} = 4$$

10. Find the limit:

$$a. \lim_{x \rightarrow 2} \frac{\sin(x-2)}{x^2-4}$$

$$c. \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

$$b. \lim_{x \rightarrow 3} \frac{\sin(x-3)}{x^3 - x^2 - 6x}$$

$$d. \lim_{x \rightarrow \pi} \frac{\sin 2x}{x^2 - \pi^2}$$

$$e. \lim_{x \rightarrow 0} \frac{x + \tan x}{3x - \sin 4x}$$

$$f. \lim_{x \rightarrow 0} \frac{4x - \sin 3x}{x + \sin(\pi x)}$$

$$g. \lim_{x \rightarrow 0} \frac{\cos 4x - \cos 3x}{x^2}$$

$$h. \lim_{x \rightarrow 0} \frac{\sin^3(4x)}{\sin(8x^3)}$$

11. Find $\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1}$, if

$$a. f(x) = x^2 + 2x$$

$$b. f(x) = \sqrt{4-x^2}$$

$$c. f(x) = x^3 + 5x$$

$$d. f(x) = e^x$$

12. Find the limit:

$$a. \lim_{x \rightarrow 2^+} \frac{x-3}{x-2}$$

$$b. \lim_{x \rightarrow 3} \frac{x^2}{x^2-9}$$

$$c. \lim_{x \rightarrow 2^-} \frac{x^2 + 2x - 3}{x^2 + x - 6}$$

$$d. \lim_{x \rightarrow 1} \frac{x^2 + x}{(x^2 + 1)(x - 1)}$$

$$e. \lim_{x \rightarrow 0^+} \frac{2}{\sin x}$$

$$f. \lim_{x \rightarrow 1^-} \frac{x^2 - 3x}{x - 1}$$

$$g. \lim_{x \rightarrow 1^+} \frac{x^2 - 3x}{x - 1}$$

$$h. \lim_{x \rightarrow 0^+} \ln x$$

$$i. \sec x$$

$$x \rightarrow \frac{\pi}{2}$$

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13. Determine the maximum possible interval on which the function is continuous.

a. $f(x) = \sqrt{x^2 - 4}$

g. $f(x) = \sqrt{\frac{x-2}{x}}$

b. $f(x) = \sqrt{9 - x^2}$

h. $f(x) = \ln\left(\frac{x-3}{e^x - 2}\right)$

c. $f(x) = \frac{x}{x^2 - 9}$

i. $f(x) = \sqrt{\ln \sqrt{x}}$

d. $f(x) = \frac{x+2}{x-3}$

j. $f(x) = \begin{cases} \frac{3|4-x^2|}{x-2} & \text{if } x \neq 2 \\ -12 & \text{if } x = 2 \end{cases}$

e. $f(x) = \sqrt{\ln(x-2)}$

k. $f(x) = \begin{cases} \frac{x^3 + 8}{x+2}, & \text{if } x \neq -2 \\ 12 & \text{if } x = -2 \end{cases}$

f. $f(x) = [x+3]$

14. Determine the value of a, b and c such that the function is continuous at each $x \in (-\infty, \infty)$.

a. $f(x) = \begin{cases} x^2 - 2, & \text{if } x \leq 2 \\ cx + 5, & \text{if } x > 2 \end{cases}$

c. $f(x) = \begin{cases} e^{ax-4} & \text{if } x \geq 0 \\ 2x + a & \text{if } x < 0 \end{cases}$

b. $f(x) = \begin{cases} x^3 + 1, & x \leq 2 \\ ax^2 + 4, & x > 2 \end{cases}$

d. $f(x) = \begin{cases} \frac{\tan(cx)}{4x}, & x \neq 0 \\ \frac{1}{c} & \text{if } x = 0 \end{cases}$



Unit Twelve

I ntroduction to Differential Calculus

What is Calculus?

Calculus is the mathematics:

- of change, velocity, acceleration
- of tangent line, slope, areas, volume

Calculus arises from four major problems these are:

- i) the tangent line problem
- ii) the velocity and acceleration problem
- iii) the minimum and maximum problem
- iv) the area problem

Introduction to Derivative

In this unit will introduce the concept of “**derivative**” which is the main tool used to calculate *rate of change* and the *slope of curve*, *slope of tangent line to curve* and etc.

Average rate of change of function

Definition: If $y = f(x)$ then the average of change of y with respect to x over the interval $[a, x_0]$ is:

- $r_{av} = \frac{\Delta y}{\Delta x} = \frac{f(x_0) - f(a)}{x_0 - a}$ = slope of secant line
- $r_{inst} = \lim_{x_0 \rightarrow a} \frac{f(x_0) - f(a)}{x_0 - a}$. . . slope of curve at $(a, f(a))$

Note: The limit of the slope of secant line of $y = f(x)$ at $x = a$ is the same as:

- slope of curve at a point $x = a$
- slope of graph of function at a point
- slope of tangent line at $(a, f(a))$
- the rate of change of y with respect to x at $x = a$
- the derivative of f at $x = a$

Illustrative Example

1. Let $y = x^2 + 3x$. Then find:

- The average rate of change of y , w. r. t. x over $[1, 3]$.
- The instantaneous rate of change of y , w. r. t. x , at $x = 1$.
- The slope of secant line passing through $(-1, -2)$ and $(2, 10)$.
- The slope of tangent line to the curve at $x = -1$.

Solution. Here, Let $y = f(x) \Rightarrow f(x) = x^2 + 3x$

$$\text{a. } r_{av} = \frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{18 - 4}{2} = 7$$

$$\begin{aligned} \text{b. } r_{inst} &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 + 3x - 4}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x-1)(x+4)}{x-1} = \lim_{x \rightarrow 1} (x+4) = 5 \end{aligned}$$

$$\text{c. } m_{sec} = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{10 - (-2)}{2 - (-1)} = \frac{12}{3} = 4$$

- d. The slope of tangent line at $x = -1$ is:

$$m_{curve} = \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x + 1} \leftarrow \text{slope of tangent line}$$

$$\Rightarrow m_{curve} = \lim_{x \rightarrow -1} \frac{x^2 + 3x + 2}{x + 1} = \lim_{x \rightarrow -1} \frac{(x+1)(x+2)}{x+1}$$

$$\Rightarrow m_{curve} = \lim_{x \rightarrow -1} (x+2) = -1 + 2 = 1$$

$\leftarrow$ slope of tangent line

Derivative as rate of change at a point

Definition: Let x_0 be a number in the domain of function f . Then, the ratio $\frac{f(x) - f(x_0)}{x - x_0}$ is called difference of quotient of f at x_0 .

Definition of derivative

Let x_0 be a number in the domain of a function f . If

$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exist we call this limit "the derivative of f at x_0 "

and write it $f'(x_0)$, so that

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exist we say f has derivative at x_0 . That is f is differentiable at x_0 .

• $f'(x_0)$ is read "the derivative of f at x_0 " or f prime of x_0 .

$f'(x_0)$ is the slope of the line tangent to the graph of f at $(x_0, f(x_0))$

Equation of tangent line

The equation of tangent line to the graph of f at $(x_0, f(x_0))$ is given by: $y - f(x_0) = f'(x_0)(x - x_0)$.

Illustrative Examples

2. Find i) the difference of quotient
 ii) the slope of tangent line at given point
 iii) the derivative of function at a given point
 iv) the equation of tangent line at the indicated point of each of the following
- | | |
|--|---|
| a. $f(x) = x^2 - x$, at $x_0 = 3$ | f. $f(x) = x^2 - 5x + 1$, at $(1, -3)$ |
| b. $f(x) = x^3 + x + 1$, at $x_0 = -2$ | g. $f(x) = \sqrt{2-x}$, at $(-7, 3)$ |
| c. $f(x) = \frac{x+1}{x+2}$, at $x_0 = 1$ | h. $f(x) = x + 3 $ at $x = -3$ |
| d. $f(x) = 2^x$, at $x = 3$ | i. $f(x) = 1 - 3x^2$; $(1, -2)$ |
| e. $f(x) = e^x$ at $x = 2$ | j. $f(x) = \cos x$ at $(\pi, -1)$ |

Solution: Here:

$$a) \quad i) \quad \frac{f(x) - f(3)}{x - 3} \rightarrow \frac{x^2 - x - 6}{x - 3} = \frac{(x - 3)(x + 2)}{x - 3}$$

$= x + 2 \leftarrow$ difference of quotient

$$ii) \quad \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} (x + 2)$$

$$= 3 + 2 = 5 \leftarrow \text{slope of tangent line}$$

$$\text{ii) } f'(3) = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3}$$

$$= \lim_{x \rightarrow 3} (x + 2) = 5 \leftarrow \text{derivative of } f \text{ at } x = 3$$

$$\text{iv) } y - f(3) = f'(3)(x - 3) \Rightarrow y - 6 = 5(x - 3)$$

$$\therefore y = 5x - 9 \leftarrow \text{Equation of tangent line at } x = 3$$

$$\text{ii) } \frac{f(x) - f(-2)}{x + 2} = \frac{x^3 + x + 1 + 9}{x + 2} = \frac{x^3 + x + 10}{x + 2} = \frac{(x + 2)(x^2 - 2x + 5)}{x + 2}$$

$$= x^2 - 2x + 5 \leftarrow \text{difference of quotient}$$

$$\text{ii) } \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2} \frac{x^3 + x + 10}{x + 2} = \lim_{x \rightarrow -2} \frac{(x + 2)(x^2 - 2x + 5)}{x + 2}$$

$$= \lim_{x \rightarrow -2} x^2 - 2x + 5 = 4 + 4 + 5 = 13 \leftarrow \text{slope of tangent line}$$

$$\text{iii) } f'(-2) = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2} x^2 - 2x + 5 = 13 \leftarrow \text{derivative}$$

$$\text{iv) } y - f(-2) = f'(-2)(x + 2) \Rightarrow y + 9 = (13)(x + 2)$$

$$\Rightarrow y = 13x + 17 \leftarrow \text{Equation of tangent line at } (-2, f(-2))$$

$$\text{ii) } \frac{f(x) - f(1)}{x - 1} = \frac{x + 1}{x + 2} - \frac{2}{3} = \frac{3x + 3 - 2x - 4}{3(x + 2)(x - 1)} = \frac{x - 1}{3(x + 2)(x - 1)}$$

$$= \frac{1}{3(x + 2)}, x \neq -2 \leftarrow \text{the difference of quotient, at } x = 1$$

$$\text{ii) } \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{1}{3(x + 2)} = \frac{1}{9} \leftarrow \text{slope of tangent line at } x = 1$$

$$\text{iii) } f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{1}{3(x + 2)} = \frac{1}{9} \leftarrow \text{derivative of } f \text{ at } x = 1$$

$$\text{iv) } y - f(1) = f'(1)(x - 1) \Rightarrow y - \frac{2}{3} = \frac{1}{9}(x - 1) \Rightarrow y = \frac{1}{9}x - \frac{5}{9}$$

$$\therefore y = \frac{1}{9}x - \frac{5}{9} \leftarrow \text{Equation of tangent line at } x = 1.$$

d) i) $\frac{f(x) - f(3)}{x - 3}$, Alternatively: let $x - 3 = h \Rightarrow x = 3 + h$

$$\Rightarrow \frac{f(3+h) - f(3)}{h} = \frac{2^{3+h} - 2^3}{h} = \frac{2^3 \cdot 2^h - 2^3}{h} = 2^3 \left(\frac{2^h - 1}{h} \right) = 8 \left(\frac{2^h - 1}{h} \right)$$

ii) $\lim_{h \rightarrow 0} 8 \left(\frac{2^h - 1}{h} \right) = 8 \lim_{h \rightarrow 0} \left(\frac{2^h - 1}{h} \right) = 8 \ln 2$

$\therefore 8 \ln 2 \leftarrow$ slope of tangent line

iii) $f'(3) = 8 \ln 2 \leftarrow$ derivative of 2^x at $x = 3$

iv) Equation of tangent line:

$$y - f(3) = f'(3)(x - 3) \Rightarrow y - 8 = 8 \ln 2 (x - 3) \Rightarrow y = 8 \ln 2 (x - 3) + 8$$

e) i) Let $x - 2 = h$. Then $\frac{f(2+h) - f(2)}{h} = \frac{e^{2+h} - e^2}{h} = e^2 \left(\frac{e^h - 1}{h} \right)$

ii) $\lim_{h \rightarrow 0} e^2 \left(\frac{e^h - 1}{h} \right) = e^2 \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = e^2 \ln(e) = e^2$

iii) $f'(2) = e^2 \leftarrow$ derivative of e^x at $x = 2$

iv) Equation of tangent line at $x = 2$ is $y - f(2) = f'(2)(x - 2)$
 $\Rightarrow y - e^2 = e^2(x - 2) \Rightarrow y = e^2x - 2e^2 + e^2 \Rightarrow y = e^2x - e^2$

f) exercise left for you

g) i) $\frac{f(x) - f(-7)}{x + 7} = \frac{\sqrt{2-x} - 3}{x + 7} \times \frac{\sqrt{2-x} + 3}{\sqrt{2-x} + 3} = \frac{-1}{\sqrt{2-x} + 3}$

ii) $\lim_{x \rightarrow -7} \frac{f(x) - f(-7)}{x + 7} = \lim_{x \rightarrow -7} \frac{1}{\sqrt{2-x} + 3} = \frac{-1}{3+3}$
 $= -\frac{1}{6}$ the slope of tangent line

iii) $f'(-7) = \lim_{x \rightarrow -7} \frac{f(x) - f(-7)}{x + 7} = -\frac{1}{6} \leftarrow$ derivative of $\sqrt{2-x}$ at $x = -7$

iv) Equation of tangent line at $x = -7$ is

$$y - f(-7) = f'(-7)(x + 7) \Rightarrow y - 3 = -\frac{1}{6}(x + 7)$$

$\Rightarrow 6y + x = 11 \leftarrow$ equation of tangent line at $(-7, f(-7))$

ii) The difference of quotient:

$$\frac{f(x) - f(-3)}{x + 3} \Rightarrow \frac{|x + 3|}{x + 3} = \begin{cases} 1 & \text{if } x > -3 \\ -1 & \text{if } x < -3 \end{cases}$$

$$\lim_{x \rightarrow -3^+} \frac{f(x) - f(-3)}{x + 3} = \lim_{x \rightarrow -3^+} \frac{|x + 3|}{x + 3} \Rightarrow \lim_{x \rightarrow -3^+} \frac{|x + 3|}{x + 3} = \frac{x + 3}{x + 3} = 1 \text{ and}$$

$$\lim_{x \rightarrow -3^-} \frac{|x + 3|}{x + 3} = \lim_{x \rightarrow -3^-} \frac{-(x + 3)}{x + 3} = -1$$

$$\Rightarrow \lim_{x \rightarrow -3} \frac{f(x) - f(-3)}{x + 3} = \lim_{x \rightarrow -3} \frac{|x + 3|}{x + 3} \text{ doesn't exist}$$

$\therefore f$ has no slope, at $x = -3$

• f has no derivative at $x = -3$

• f has no tangent line equation

Exercise left for you.

$$\text{ii) } \frac{f(x) - f(\pi)}{x - \pi} = \frac{\cos x + 1}{x - \pi}, \text{ let } x - \pi = h \Rightarrow x = \pi + h$$

$$\Rightarrow \frac{\cos(\pi + h) + 1}{h} = \frac{\cos \pi \cos h - \sin(\pi) \sin h + 1}{h}$$

$$\Rightarrow \frac{-\cos h + 1}{h} = \frac{1 - \cos h}{h}, h \neq 0$$

$$\text{ii) } \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0 \leftarrow \text{slope of tangent line at } x = \pi$$

$$\text{iii) } f'(\pi) = 0$$

iv) Equation of tangent line at $x = \pi$

$$\Rightarrow y - f(\pi) = f'(\pi)(x - \pi) \Rightarrow y - (-1) = 0(x - \pi)$$

$$\Rightarrow y + 1 = 0 \Rightarrow y = -1$$

Finding slope and equation of tangent line of the pieces wisely defined function

Illustrative Example

3. Using formal definition (limit concept). Find the equation of the line tangent to the graph of f at the indicated point for the following function:

a. $f(x) = \begin{cases} x^2 + 2 & \text{if } x \leq 2 \\ 4x - 2 & \text{if } x > 2 \end{cases}$ at $P(2, 6)$

b. $f(x) = \begin{cases} -2x + 9 & \text{if } x \geq 1 \\ 8 - x^2 & \text{if } x < 1 \end{cases}$ at $P(1, 7)$

c. $f(x) = \begin{cases} 4x^2 + 7x & \text{if } x \leq -3 \\ 3x + 24 & \text{if } x > -3 \end{cases}$ at $P(-3, 15)$

Solution: First determine the slope of the curve at given point.

a) $f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$

Note:

- Right side limit implies right side derivative of f , write $f'_+(a)$.
- Left side limit implies left side derivative of f write $f'_-(a)$.

$$\Rightarrow f'_+(2) = \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{4x - 2 - 6}{x - 2} = \frac{4(x - 2)}{x - 2} = 4$$

$$\begin{aligned} f'_-(2) &= \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{x^2 + 2 - 6}{x - 2} \\ &= \lim_{x \rightarrow 2^-} \frac{(x - 2)(x + 2)}{x - 2} = 4 \end{aligned}$$

$\therefore f'(2) = 4 \leftarrow$ slope of tangent line

$$y - f(2) = f'(2)(x - 2) \Rightarrow y - 6 = 4(x - 2)$$

$$\Rightarrow y = 4x - 2 \leftarrow \text{equation of tangent line}$$

b) $f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1}$ provided that the limit

$$= \lim_{x \rightarrow 1^+} \frac{-2(x-1)}{x-1} = -2 \text{ (right side limit) and}$$

$$\lim_{x \rightarrow 1^-} \frac{8-x^2-7}{x-1} = \lim_{x \rightarrow 1^-} \frac{(1-x)(1+x)}{x-1} = -2 \text{ (left side limit).}$$

$$\text{Therefore } \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x-1} = f'(1) = -2.$$

Hence, the equation of tangent line $y - f(1) = f'(1)(x - 1)$

$$y - 7 = -2(x - 1) \Rightarrow y = -2x + 9$$

c) Exercise left for you.

Let $f(x) = \begin{cases} 4x & \text{if } x \geq 3 \\ x^2 + 3 & \text{if } x < 3 \end{cases}$ find the equation of the tangent

line to the graph of f at the following point.

a. $(-1, 4)$ b. $(3, 12)$ c. $(4, 16)$

Solution: Here

$$a. f'(-1) = \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{x^2 + 3 - 4}{x + 1} = \lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1}$$

$$\Rightarrow f'(-1) = \lim_{x \rightarrow -1} \frac{(x-1)(x+1)}{x+1} = -1 - 1 = -2 \quad \leftarrow \text{slope}$$

$\therefore$ The equation of tangent line: $y - f(-1) = f'(-1)(x - (-1)) \Rightarrow$

$$y - 4 = -2(x + 1) \Rightarrow y = -2x + 2$$

$$b. f'(3) = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3}$$

$$f_+(3) = \lim_{x \rightarrow 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3^+} \frac{4x - 12}{x - 3} = \frac{4(x-3)}{x-3} = 4$$

$$\text{and } f_-(3) = \lim_{x \rightarrow 3^-} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3^-} \frac{x^2 + 3 - 12}{x - 3} = \lim_{x \rightarrow 3^-} \frac{x^2 - 9}{x - 3}$$

$$= \lim_{x \rightarrow 3^-} \frac{(x-3)(x+3)}{x-3} = 3 + 3 = 6$$

Since the right side and the left side limit of f is not equal.

$$\Rightarrow \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} \text{ does not exist.}$$

$\Rightarrow f'(3)$ does not exist.

Therefore, the graph of f has no slope and has no tangent line at $(3, 12)$.

$$\text{c. } f'(4) = \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \rightarrow 4} \frac{4x - 16}{x - 4} = \lim_{x \rightarrow 4} \frac{4(x - 4)}{x - 4} = 4$$

$\Rightarrow$ Equation of tangent line will be $y - f(4) = f'(4)(x - 4)$

$$\Rightarrow y - 16 = 4(x - 4) \Rightarrow y = 4x$$

5. If $f(x) = x^3 + x^2 - 4x - 4$ then find the equation of the tangent line at the point where $f(x)$ crosses:

a. the x -axis b. the y axis c. the graph of $y = x^2 - 5x + 4$

Solution: Let f has slope/ derivative at a , so that:

$$\begin{aligned} \Rightarrow f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x^3 + x^2 - 4x - 4) - (a^3 + a^2 - 4a - 4)}{x - a} \end{aligned}$$

$$\Rightarrow f'(a) = \lim_{x \rightarrow a} \frac{x^3 - a^3 + x^2 - a^2 - 4x + 4a - 4 + 4}{x - a}$$

$$\Rightarrow f'(a) = \lim_{x \rightarrow a} \frac{(x - a)(x^2 + ax + a^2) + (x - a) - (x + a) - 4(x - a)}{x - a}$$

$$\Rightarrow f'(a) = \lim_{x \rightarrow a} (x^2 + ax + a^2 + x + a - 4) = 3a^2 + 2a - 4$$

Then:

a. The graph of $f(x) = x^3 + x^2 - 4x - 4$ meets the x -axis when $y = f(x) = 0$.

$$\Rightarrow x^3 + x^2 - 4x - 4 = 0 \Rightarrow x^2(x + 1) - 4(x + 1) = 0$$

$$\Rightarrow (x^2 - 4)(x + 1) = 0 \Rightarrow x = 2, -2, -1$$

$\therefore$ The graph intersect the x -axis at $x = -2, 2, -1$.

$$\text{Since, } f'(a) = 3a^2 + 2a - 4, f'(-2) = 3(-2)^2 + 2(-2) - 4 = 4$$

$\therefore$ slope at $x = -2$ is 4.

$$\therefore y - f(-2) = f'(-2)(x + 2) \Rightarrow y - 0 = 4(x + 2)$$

$\therefore$ The equation of tangent line $y = 4x + 8$.

$$\Rightarrow f'(2) = 3(2)^2 + 2(2) - 4 = 12 \leftarrow \text{slope at } x = 2$$

$$y - f(2) = f'(2)(x - 2) \Rightarrow y = 12(x - 2) = 12x - 24$$

Introduction to Differential Calculus

Equation of tangent line at $x = 2$ is $y = 12x - 24$.

Equation of tangent line at $x = -1$ is $y - f(-1) = f'(-1)(x - (-1)) \Rightarrow y = -3(x + 1)$

The graph of $f(x) = x^3 + x^2 - 4x - 4$ meets the y -axis when $x = 0$

$$\Rightarrow f(0) = 0 + 0 - 0 - 4 = -4, f'(0) = 3(0)^2 + 2(0) - 4 = -4$$

Equation of tangent line at $x = 0$ is $y - f(0) = f'(0)(x - 0) \Rightarrow y = -4x - 4$

The intersection of $y = x^3 + x^2 - 4x - 4$ and $y = x^2 + 5x - 4$

$$\Rightarrow x^3 + x^2 - 4x - 4 = x^2 + 5x - 4 \Rightarrow x^3 - 9x = 0$$

$$\Rightarrow x(x^2 - 9) = 0 \Rightarrow x = 0 \text{ and } x = \pm 3$$

The equation of tangent line at point $(0, -4)$, $(3, 20)$ and $(-3, -10)$ is given by:

• At $x = 0 \Rightarrow y - f(0) = f'(0)(x - 0) \Rightarrow y + 4 = -4x \Rightarrow y = -4x - 4$

• At $x = 3 \Rightarrow y - f(3) = f'(3)(x - 3) \Rightarrow y - 20 = 17(x - 3)$

Explanation: $f'(-3) = 3(-3)^2 + 2(-3) - 4 = 17$

• At $x = -3 \leftarrow$ Exercise left for you.

For what value of x , is pairs of each of the following functions have the same slope.

a. $f(x) = -x^3 + 4x$ and $g(x) = \frac{1}{2}x^2 + 6$

b. $f(x) = x^3 - 8x$ and $g(x) = 4 - x^2$

c. $f(x) = x^2 + 5x - 4$ and $g(x) = x^3 + x^2 - 4x - 4$

d. $f(x) = 1 - x^2$ and $g(x) = x^3 - x^2 - x + 1$

Solution: If $f(x)$ and $g(x)$ have the same slope then $f'(x) = g'(x)$, so that:

a. $f'(x) = g'(x) \Rightarrow -3x^2 + 4 = x \Rightarrow 3x^2 + x - 4 = 0$

$$\Rightarrow (3x + 4)(x - 1) = 0 \Rightarrow x = -\frac{4}{3} \text{ and } x = 1$$

b. $f'(x) = g'(x) \Rightarrow 3x^2 - 8 = -2x \Rightarrow 3x^2 + 2x - 8 = 0$

$$\Rightarrow (3x - 4)(x + 2) = 0 \Rightarrow x = \frac{4}{3} \text{ and } x = -2$$

c. $f'(x) = g'(x) \Rightarrow 2x + 5 = 3x^2 + 2x - 4 \Rightarrow 3x^2 = 9$

$$\Rightarrow x = -\sqrt{3} \text{ or } x = \sqrt{3}$$

d. Exercise left for you.

Relation between limit and derivative

Illustrative example

7. Let $f(x) = x^5 + 6x + 4$. Find

a. $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

c. $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$

b. $\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$

Solution: Here

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \leftarrow \text{Rate change of } y \text{ w. r. t. } x$$

$$f(x) = x^5 + 6x + 4 \Rightarrow f'(x) = 5x^4 + 6$$

a. $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x) = 5x^4 + 6$

b. $f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = 5(1)^4 + 6 = 11$

c. $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$ is the same as $f'(2)$, so that

$$\Rightarrow f'(2) = 5(2)^4 + 6 = 86$$

8. Let $f(x) = x^3 - 6x$. Then find the set of values of x for which the tangent line has:

a. slope 0

c. slope positive

b. slope 3

d. slope negative

Solution: Use $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$, thus $f(x) = x^3 - 6x$
 $\Rightarrow f'(x) = 3x^2 - 6$, so that

a. $f'(x) = 0 \Rightarrow 3x^2 - 6 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2} \leftarrow \text{Horizontal tangent line}$

b. $f'(x) = 3 \Rightarrow 3x^2 - 6 = 3 \Rightarrow 3x^2 = 9 \Rightarrow x^2 = 3 \therefore x = \pm\sqrt{3}$

c. slope positive means $f'(x) > 0 \Rightarrow 3x^2 - 6 > 0$

$$\Rightarrow 3(x^2 - 2) > 0 \Leftrightarrow 3(x - \sqrt{2})(x + \sqrt{2}) > 0$$

 $\therefore$ for all $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ the slope is positive.

d. Slope negative means that at each x , such that $f'(x) < 0$

$$\Rightarrow 3x^2 - 6 < 0 \Rightarrow 3(x^2 - 2) < 0 \Rightarrow 3(x - \sqrt{2})(x + \sqrt{2}) < 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^{\frac{1}{3}}}{x} = \lim_{x \rightarrow 0} \frac{1}{x^{\frac{2}{3}}} = \infty$$

$\therefore f$ has vertical tangent line at $x = 0$.

Note: f is not differentiable at $x = 0$ since the limit doesn't exist.

b) $f(x) = (x-2)^{\frac{1}{3}}$, f is cont. at $x = 2$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)^{\frac{1}{3}} - 0}{x - 2} = \lim_{x \rightarrow 2} \frac{1}{(x-2)^{\frac{2}{3}}} = \infty$$

$\therefore f$ has vertical tangent line at $x = 2$

Note: f has no derivative at $x = 2$.

c) $f(x) = x^{\frac{4}{5}}$, f is cont. at $x = 0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^{\frac{4}{5}}}{x} = \lim_{x \rightarrow 0} \frac{1}{x^{\frac{1}{5}}}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{1}{x^{\frac{1}{5}}} = \infty \text{ and } \lim_{x \rightarrow 0^-} \frac{1}{x^{\frac{1}{5}}} = -\infty$$

Since both side of the limit is not the same sign: $f(x) = x^{\frac{4}{5}}$ has no vertical tangent line. (the graph has cusp shape and corner).

d) f is cont. at $x = 3$. However, the one sided limit

$$\Rightarrow \lim_{x \rightarrow 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3^+} \frac{|x-3|}{x-3} = 1$$

$$\Rightarrow \lim_{x \rightarrow 3^-} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3^-} \frac{|x-3|}{x-3} = -1$$

Derivatives from the right and from the left are not equal and (∞) . So f has no tangent line at $x = 3$.

$\Rightarrow f$ has no derivative at $x = 3$

e) f is cont. from the right at $x = 2$.

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$$\Rightarrow \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{\sqrt{x-2}}{x-2} = \lim_{x \rightarrow 2^+} \frac{1}{\sqrt{x-2}} = \infty$$

$\therefore f$ has vertical tangent line at $x = 2$.

i) f is cont. at $x = 0$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^{\frac{2}{3}}}{x} = \lim_{x \rightarrow 0} \frac{1}{x^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{1}{x^{\frac{1}{3}}} = \infty \text{ and } \lim_{x \rightarrow 0^-} \frac{1}{x^{\frac{1}{3}}} = -\infty$$

The one sided limits are not the same sign.

$\therefore f$ has no vertical tangent line at $x = 0$.

Derivative as a function

The domain of f' is the set of all x at which f is differentiable and whose value at any number x is given by:

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \text{ is called the derivative of } f \text{ at } x.$$

Illustrative Example

12. Using limit concept, find the derivative of function with respect to x .

a. $f(x) = -5x^2 + x$

d. $f(x) = \sqrt{x}$

b. $f(x) = \frac{1}{x+1}, x \neq -1$

e. $f(x) = \frac{x+2}{4-3x}$

c. $f(x) = x^3 - 4$

f. $f(x) = a^x$

Solution: Here

$$\begin{aligned} \text{a) } f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x} \frac{-5t^2 + t - (-5x^2 + x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{5x^2 - 5t^2}{t - x} + \lim_{t \rightarrow x} \frac{t - x}{t - x} = \lim_{t \rightarrow x} \frac{5(x-t)(t+x) + 1}{t - x} = -10x + 1 \\ &\Rightarrow f(x) = -5x^2 + x \Rightarrow f'(x) = -10x + 1 \end{aligned}$$

$$\begin{aligned}
 \text{b) } f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x} \frac{\frac{1}{t+1} - \frac{1}{x+1}}{t - x} \\
 &= \lim_{t \rightarrow x} \frac{x+1 - t-1}{(t-x)(t+1)(x+1)} = \lim_{t \rightarrow x} \frac{x-t}{(t-x)(t+1)(x+1)} \\
 &= -\lim_{t \rightarrow x} \frac{1}{(t+1)(x+1)} = -\frac{1}{(x+1)^2} \text{ for } x \neq -1 \\
 &\Rightarrow f(x) = \frac{1}{x+1} \Rightarrow f'(x) = \frac{-1}{(x+1)^2} \text{ for } x \neq -1
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } f'(x) &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x} \frac{(t^3 - 4) - (x^3 - 4)}{t - x} \\
 &= \lim_{t \rightarrow x} \frac{t^3 - x^3}{t - x} = \lim_{t \rightarrow x} \frac{(t-x)(t^2 + tx + x^2)}{t - x} = \lim_{t \rightarrow x} (t^2 + tx + x^2) = 3x^2
 \end{aligned}$$

$$\Rightarrow f(x) = x^3 - 4 \Rightarrow f'(x) = 3x^2$$

$$\text{d) } f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = \lim_{t \rightarrow x} \frac{\sqrt{t} - \sqrt{x}}{t - x} = \lim_{t \rightarrow x} \frac{\sqrt{t} - \sqrt{x}}{t - x} \cdot \frac{\sqrt{t} + \sqrt{x}}{\sqrt{t} + \sqrt{x}}$$

$$\Rightarrow f'(x) = \lim_{t \rightarrow x} \frac{t - x}{t - x} \lim_{t \rightarrow x} \frac{1}{\sqrt{t} + \sqrt{x}} = \frac{1}{2\sqrt{x}}, \text{ for } x > 0$$

e) exercise left for you

12b. Let $f(x+y) = f(x)f(y)$ for all real x and y . If $f(4) = 3$ and $f'(0) = 4$, then what is the value of $f'(4)$?

A. 12 B. 1 C. 7 D. 0

Explanation: Here

$$f'(4) = \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0} \frac{f(4+h) - f(4+0)}{h}$$

$$\text{Since } f(x+y) = f(x)f(y) \Rightarrow f(4+h) = f(4)f(h)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(4)f(h) - f(4)f(0)}{h} = \lim_{h \rightarrow 0} f(4) \left[\frac{f(h) - f(0)}{h} \right]$$

$$\Rightarrow f(4) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = f(4)f'(0) = (3)(4) = 12$$

Answer: A

Differentiable Function

If f is differentiable at each number in its domain, then f is differentiable.

Differentiability of function at a point a . We say a function f is

differentiable at point $x = a$ if $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exist.

Differentiability over an interval.

i) We say a function f is differentiable on (a, b) if f is differentiable at all number inside the interval (a, b) i.e., $a < x_0$

$$< b \Rightarrow f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exist.}$$

ii) We say a function f is differentiable on $[a, b]$ if f is differentiable at all number inside the interval (a, b) and if the

one side limit, $\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}$ exist and

$$\lim_{x \rightarrow b^-} \frac{f(x) - f(b)}{x - b} \text{ exist.}$$

A function f is differentiable on $[a, \infty)$ if f is differentiable at all

number in (a, ∞) and $\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}$ exist.

A function f is differentiable on $(-\infty, a]$ if f is differentiable at all

number inside the interval $(-\infty, a)$ and $\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a}$ exist.

Note: Non differentiability occurs where the graph of f has:

- Sharp corner (especially at the zeros of absolute value).
- Vertical tangent line.
- Discontinuity

Illustrative Example

For each of the following functions, find the set of values of x_0 , such that:

- i) $f'(x_0)$ does not exist.
- ii) $f'(x_0)$ exist

- a. $f(x) = |x|$ c. $f(x) = |x^2 - 4|$ e. $f(x) = \sqrt{x-4}$
 b. $f(x) = |x+3|$ d. $f(x) = \sqrt{x}$ f. $f(x) = [x]$
 g. $f(x) = (x-1)^{\frac{2}{3}}$ h. $f(x) = \frac{x+2}{3-x}$ i. $f(x) = |x| - x$

Solution: x_0 is likely or possibly is the zeros of f and point of discontinuity.

$$\text{a. } \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x} \Rightarrow \lim_{x \rightarrow 0^+} \frac{x}{x} = 1 \text{ and } \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$$

$\Rightarrow$ The right and the left side limits are not equal. Therefore:

- (i) $f'(0)$ does not exist (that is f is not differentiable at $x_0 = 0$)
 (ii) f is differentiable on $(-\infty, 0]$ and $[0, \infty)$

$$\text{b. i) } \lim_{x \rightarrow -3} \frac{f(x) - f(-3)}{x + 3} \Rightarrow \lim_{x \rightarrow -3^+} \frac{|x+3|}{x+3} = \lim_{x \rightarrow -3^+} \frac{x+3}{x+3} = 1 \text{ and}$$

$$\lim_{x \rightarrow -3^-} \frac{|x+3|}{x+3} = \lim_{x \rightarrow -3^-} \frac{-(x+3)}{x+3} = -1 \Rightarrow \lim_{x \rightarrow -3} \frac{f(x) - f(-3)}{x + 3} \text{ does not exist}$$

$\Rightarrow f'(-3)$ does not exist.

$\Rightarrow f$ is not differentiable at $x_0 = -3$

- ii) f is differentiable on $(-\infty, -3]$ and $[-3, \infty)$

$$\text{c. i) } f(x) = |x^2 - 4| \Rightarrow f'(x) = \begin{cases} 2x, & \text{if } x \leq -2 \text{ or } x \geq 2 \\ -2x & \text{if } -2 < x < 2 \end{cases}$$

$\Rightarrow f'(-2)$ and $f'(2)$ does not exist.

- ii) f is differentiable on $(-\infty, -2]$, $[-2, 2]$ and $[2, \infty)$.

- d. Domain of $f = \{x: x \geq 0\}$

$$\text{i) } \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{x} = \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{x}} = \infty$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} \text{ does not exist } \Rightarrow f'(0) \text{ does not exist.}$$

$\Rightarrow f$ is not differentiable at $x_0 = 0$.

- ii) f is differentiable on $(0, \infty)$.

- e. Domain of $f = \{x: x \geq 4\}$

$$c) \lim_{x \rightarrow 4^+} \frac{f(x) - f(4)}{x - 4} = \lim_{x \rightarrow 4^+} \frac{\sqrt{x-4}}{x-4} = \lim_{x \rightarrow 4^+} \frac{1}{\sqrt{x-4}} = \infty$$

$$\Rightarrow \lim_{x \rightarrow 4^+} \frac{f(x) - f(4)}{x - 4} \text{ does not exist} \Rightarrow f'(4) \text{ does not exist}$$

$\Rightarrow f$ is not differentiable at $x_0 = 4$

ii) f is differentiable on $(4, \infty)$

i) Since f is not continuous on integer $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$\Rightarrow f$ is not differentiable on integer $\mathbb{Z}$.

$$\bullet \lim_{x \rightarrow k^+} \frac{f(x) - f(k)}{x - k} = 0 \leftarrow \text{Differentiable from the right}$$

$$\bullet \lim_{x \rightarrow k^-} \frac{f(x) - f(k)}{x - k} = \infty \leftarrow \text{Not differentiable from the left}$$

$$g) i) \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{(x-1)^{\frac{2}{3}}}{x-1} = \lim_{x \rightarrow 1^+} \frac{1}{(x-1)^{\frac{1}{3}}} = \infty \text{ and}$$

$$\lim_{x \rightarrow 1^-} \frac{1}{(x-1)^{\frac{1}{3}}} = -\infty \Rightarrow f'(1) \text{ does not exist} \Rightarrow f \text{ is not differentiable at } x=1$$

ii) f is differentiable on $(-\infty, 1)$ and $(1, \infty)$

h. Since f is not continuous at $x=3$.

i) $f'(3)$ does not exist

ii) f is differentiable at on $\mathbb{R} \setminus \{3\}$

$$i. \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = 0 \text{ and } \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{-x - x}{x - 0} = \lim_{x \rightarrow 0^-} \frac{-2x}{x} = -2 \Rightarrow i) \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} \text{ does not exist}$$

$\Rightarrow f'(0)$ does not exist $\Rightarrow f$ is not differentiable at $x=0$

ii) f is differentiable on $(-\infty, 0]$ and $[0, \infty)$



One side differentiability

i) The right side differentiability of f at x_0 is defined by

$$f'_+(x) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \text{ provided the limit exist.}$$

Equivalently:

$$f'_+(x_0) = \lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \text{ provided the limit exist.}$$

ii) The left side differentiability of f at x_0 is defined by:

$$f'_-(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \text{ provided the limit exist.}$$

Equivalently:

$$f'_-(x_0) = \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \text{ provided the limit exist.}$$

Clearly $f'(x_0)$ exist if and only if $f'_+(x_0)$ and $f'_-(x_0)$ both exist and are equal.

Conceptual Example

14. Let $f(x)$ be continuous and differentiable at $x = 3$. If $f(3) = 7$ and

$$f'(3) = 1 \text{ then what is the value of } \lim_{x \rightarrow 3} \frac{xf(3) - 3f(x)}{x - 3}.$$

A. 10 B. 6 C. 5 D. 4

Explanation: Given that $f(3) = 7$, $f'(3) = 1$.

Now add the expression: $-3f(3) + 3f(3)$.

$$\text{Here, } \lim_{x \rightarrow 3} \frac{xf(3) - 3f(x)}{x - 3} = \lim_{x \rightarrow 3} \frac{xf(3) - 3f(3) + 3f(3) - 3f(x)}{x - 3}$$

$$\Rightarrow \lim_{x \rightarrow 3} \frac{f(3)(x - 3) - 3(f(x) - f(3))}{x - 3} = \lim_{x \rightarrow 3} \frac{f(3)(x - 3)}{x - 3} - 3 \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3}$$

$$\Rightarrow \lim_{x \rightarrow 3} f(3) - 3f'(3) = 7 - 3(1) = 4$$

Answer: D

If $f(x)$ and $g(x)$ be two functional such that $f(a) = 3, f'(a) = 2, g(a) = -2$ and $g'(a) = -3$, then what is the value of

$$\lim_{x \rightarrow a} \frac{g(x)f(a) - g(a)f(x)}{x - a}$$

A. 13 B. -5 C. 6 D. -6

Explanation:

Here, $f(a) = 3, f'(a) = 2, g(a) = -2, g'(a) = -3$

Now, add the expression : $-g(a)f(a) + g(a)f(a)$

$$\text{Then, } \lim_{x \rightarrow a} \frac{g(x)f(a) - g(a)f(x)}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{g(x)f(a) - g(a)f(a) + g(a)f(a) - g(a)f(x)}{x - a}$$

$$= \lim_{x \rightarrow a} \left[f(a) \frac{g(x) - g(a)}{x - a} - g(a) \frac{f(x) - f(a)}{x - a} \right]$$

$$= f(a) \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} - g(a) \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$= f(a)g'(a) - g(a)f'(a) = 3(-3) - (-2)(2) = -5$$

Answer: B

Finding Differentiability interval involving radicals

Illustrative Examples

Find the largest intervals on which each of the following function is differentiable.

$$f(x) = \sqrt{x-2}$$

d. $f(x) = \sqrt{25-16x^2}$

$$f(x) = \sqrt{3-x}$$

e. $f(x) = \sqrt{x^2-5x+6}$

$$f(x) = \sqrt{4-x^2}$$

f. $f(x) = \sqrt[3]{x-2}$

Solution:

Recall that: Domain of f' is the set of all x in the domain of f on which f is differentiable.

$$f(x) = \sqrt{x-2}, \text{ i) Domain of } f = \{x: x-2 \geq 0\} = [2, \infty)$$

$$\text{ii) } \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{\sqrt{x-2}}{x-2} = \lim_{x \rightarrow 2^+} \frac{1}{\sqrt{x-2}} = \infty$$

Hence, f is not differentiable at $x = 2$ from the right.

$\therefore$ The domain of f' is $(2, \infty)$.

$\therefore$ The largest interval on which f is differentiable is $(2, \infty)$ but f is not differentiable on $[2, \infty)$.

- b. $f(x) = \sqrt{3-x}$, i) Domain of $f = \{x: 3-x \geq 0\} = (-\infty, 3]$
 ii) Domain of $f' = \{x: x < 3\} = (-\infty, 3)$.

Since $\lim_{x \rightarrow 3^-} \frac{f(x) - f(3)}{x - 3}$ does not exist $f'_-(3)$ does not exist.

$\therefore$ The set of all x on which f is differentiable is $(-\infty, 3)$ but f is not differentiable on $(-\infty, 3]$.

- c. $f(x) = \sqrt{4-x^2}$
 i) Domain of $f = \{x: 4-x^2 \geq 0\} = \{x: -2 \leq x \leq 2\}$

- ii) Since $\lim_{x \rightarrow -2^+} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2^+} \frac{\sqrt{4-x^2}}{x+2} = \lim_{x \rightarrow -2^+} \frac{2-x}{\sqrt{4-x^2}}$ does

not exist (∞). Therefore $f'_+(-2)$ does not exist.

Similarly, $\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2}$ does not exist.

$\therefore$ The domain of $f' = \{x: -2 < x < 2\}$, that is the set of all x on which f is differentiable is $\{x: -2 < x < 2\}$

- d. $f(x) = \sqrt{25-16x^2}$

- i) Domain of $f = \left\{x: -\frac{5}{4} \leq x \leq \frac{5}{4}\right\}$

- ii) Since $\lim_{x \rightarrow -\frac{5}{4}^+} \frac{f(x) - f\left(-\frac{5}{4}\right)}{x + \frac{5}{4}} = \infty$ and $\lim_{x \rightarrow \frac{5}{4}^-} \frac{f(x) - f\left(\frac{5}{4}\right)}{x - \frac{5}{4}} = \infty$

$\Rightarrow f'_+\left(-\frac{5}{4}\right)$ and $f'_-\left(\frac{5}{4}\right)$ does not exist.

$\therefore$ The set of all x on which f is differentiable

$\left\{x: -\frac{5}{4} < x < \frac{5}{4}\right\}$, that is the domain of f' .

$$f(x) = \sqrt{x^2 - 5x + 6}$$

i) Domain of $f = \{x: (x-3)(x-2) \geq 0\} \Rightarrow \{x: x \leq 2 \text{ or } x \geq 3\}$

ii) Since $\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \infty$ and $\lim_{x \rightarrow 3^+} \frac{f(x) - f(3)}{x - 3} = \infty$

$\Rightarrow f'_-(2)$ and $f'_+(3)$ does not exist.

$\therefore$ The set of all x on which f is differentiable is $(-\infty, 2)$ and $(3, \infty)$ which is the domain of f' .

$f(x) = \sqrt[3]{x-2}$, i) Domain of $f = \{x: x \in (-\infty, \infty)\}$

ii) Since $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{1}{\sqrt[3]{(x-2)^2}} = \infty \Rightarrow f'(2)$ does not exist.

$\therefore$ The set of all x on which f is differentiable is $\mathbb{R} \setminus \{2\}$ that is $(-\infty, 2)$ and $(2, \infty)$ (domain of f')

$$f(x) = \sqrt{x+3}$$

$$f(x) = \sqrt{4-9x^2}$$

$$f(x) = \sqrt[3]{x+4}$$

$\left. \begin{array}{l} f(x) = \sqrt{x+3} \\ f(x) = \sqrt{4-9x^2} \\ f(x) = \sqrt[3]{x+4} \end{array} \right\} \leftarrow \text{Exercise left for you.}$

Finding differentiability interval involving absolute value function and logarithm function

Illustrative Example

Find i) the domain of function

ii) the largest interval on which f is differentiable (domain of f')

a. $f(x) = x|x|$

e. $f(x) = |x-1| + |x+2|$

b. $f(x) = x|x+2|$

f. $f(x) = \frac{x}{x^2-9}$

c. $f(x) = |2x+3|$

g. $f(x) = \frac{x+1}{(x-3)(x+2)}$

d. $f(x) = |x^2-4|$

h. $f(x) = \log_3 x^2$

Solution: Note the domain of absolute value is $\mathbb{R}$.

$$\text{a. } f(x) = x|x|, \text{ i) Domain of } f \text{ is } (-\infty, \infty) \Rightarrow x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

$$\text{so that } \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{x^2}{x} = 0$$

Similarly,

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{-x^2}{x} = 0 \Rightarrow f'_+(0) = f'_-(0) = 0 \Rightarrow f'(0)$$

exist.

$\therefore$ The set of all x on which f is differentiable is $(-\infty, \infty)$ which is the domain of f' .

$$\Rightarrow f'(x) = \begin{cases} 2x, & x > 0 \\ 0, & x = 0 \\ -2x, & x < 0 \end{cases} \Rightarrow f'(x) = 2|x|, \text{ for all } x \in (-\infty, \infty)$$

b. $f(x) = x|x+2|$, i) Domain of f is $(-\infty, \infty)$

$$\Rightarrow f(x) = x|x+2| = \begin{cases} x^2 + 2x, & x \geq -2 \\ -x^2 - 2x, & x < -2 \end{cases}$$

$$\text{ii) } \lim_{x \rightarrow -2^+} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2^+} \frac{x^2 + 2x}{x + 2} = \lim_{x \rightarrow -2^+} \frac{x(x+2)}{x+2} = -2$$

and

$$\lim_{x \rightarrow -2^-} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2^-} \frac{-x^2 - 2x}{x + 2} = \lim_{x \rightarrow -2^-} \frac{-x(x+2)}{x+2} = 2$$

$$\Rightarrow f'_+(-2) = -2 \text{ and } f'_-(-2) = 2$$

Hence $f'(-2)$ does not exist.

$\therefore$ The set of all x on which f is differentiable is $(-\infty, -2)$ and $(-2, \infty)$ which is domain of f' .

c. $f(x) = |2x + 3|$, i) Domain of $f = \{x: x \in (-\infty, \infty)\}$

$$\Rightarrow f(x) = |2x+3| = \begin{cases} 2x+3, & \text{if } x \geq -\frac{3}{2} \\ -2x-3, & \text{if } x < -\frac{3}{2} \end{cases}$$

$$\text{ii) } \lim_{x \rightarrow -\frac{3}{2}^+} \frac{f(x) - f\left(-\frac{3}{2}\right)}{x + \frac{3}{2}} = 2 \text{ and } \lim_{x \rightarrow -\frac{3}{2}^-} \frac{f(x) - f\left(-\frac{3}{2}\right)}{x + \frac{3}{2}} = -2$$

$$\Rightarrow f'_+\left(-\frac{3}{2}\right) = 2 \text{ and } \Rightarrow f'_-\left(-\frac{3}{2}\right) = -2$$

Hence, $f'\left(-\frac{3}{2}\right)$ does not exist.

$\therefore$ The set of all x on which f is differentiable is

$$\left(-\infty, -\frac{3}{2}\right) \text{ and } \left(-\frac{3}{2}, \infty\right).$$

d. $f(x) = |x^2 - 4| \leftarrow$ Exercise left for you.

e. $f(x) = |x-1| + |x+2|$, i) Domain of $f = \{x: x \in (-\infty, \infty)\}$

$$\Rightarrow |x-1| = \begin{cases} x-1, & x \geq 1 \\ -x+1, & x < 1 \end{cases} \text{ and } |x+2| = \begin{cases} x+2, & x \geq -2 \\ -x-2, & x < -2 \end{cases}$$

	$-\infty$	-2	1	∞
$ x-1 $		$-x+1$	$-x+1$	$x-1$
$ x+2 $		$-x-2$	$x+2$	$x+2$
$ x-1 + x+2 $		$-2x-1$	3	$2x+1$

$$f(x) = |x-1| + |x+2| = \begin{cases} -2x-1 & \text{if } x < -2 \\ 3, & \text{if } -2 \leq x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases}$$

$$\Rightarrow \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2} \frac{-2x-1-3}{x+2} = \lim_{x \rightarrow -2} \frac{-2(x+2)}{x+2} = -2$$

$$\text{Similarly, } \lim_{x \rightarrow -2^+} \frac{f(x) - f(-2)}{x + 2} = \frac{3-3}{x+2} = 0$$

$$\Rightarrow f'(-2) = -2 \text{ and } f'_+(-2) = 0 \Rightarrow f'(-2) \text{ does not exist.}$$

$$\text{Also } \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = 0 \text{ and } \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{2x - 1 - 3}{x - 1}$$

$$\Rightarrow \lim_{x \rightarrow 1^+} \frac{2(x-1)}{x-1} = 2$$

$$\text{Hence } f'_-(1) = 0 \text{ and } f'_+(1) = 2 \Rightarrow f'(1) \text{ does not exist.}$$

$\therefore f$ is differentiable on $(-\infty, -2]$, $[-2, 1]$ and $[1, \infty)$, but $f'(-2)$ and $f'(1)$ does not exist.

f. $f(x) = \frac{x}{x^2 - 9}$ i) Domain of $f = \{x: x \neq -3, 3\}$

ii) Domain of f' is $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$

g. Domain = $\{x: x \neq 3, \text{ and } x \neq -2\}$.

Domain of f' is $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

h. $f(x) = \log_3^{4-x^2}$

i) Domain of $f = \{x: -2 < x < 2\}$

ii) Domain of $f' = \{-2 < x < 2\}$

Differentiability and Continuity

The relationship between continuity and differentiability is as follows:

i) If a function f is differentiable at $x = a$, then f is continuous at $x = a$.

$\Rightarrow$ Differentiability implies continuity.

ii) If a function f is continuous at $x = a$ then f may not be differentiable at $x = a$. So continuity does not imply differentiability.

Illustrative Examples

16. Show that each of the following function is continuous but not differentiable at the indicated point.

a. $f(x) = |x - 1|$, at $x = 1$

c. $f(x) = \sqrt{x - 2}$, at $x = 2$

b. $f(x) = |2x + 3|$; $x = -\frac{3}{2}$

d. $f(x) = x^2 \sin \frac{1}{x}$, at $x = 0$

e. $f(x) = \begin{cases} x^2, & \text{if } x \leq 2 \\ 8 - 2x, & \text{if } x > 2 \end{cases}$ at $x = 2$

Solution:a) i) To show continuity at $x = 1$

$$\lim_{x \rightarrow 1} f(x) = f(1) \Rightarrow \lim_{x \rightarrow 1} |x - 1| = 0 \Rightarrow f(1) = 0 \Rightarrow f \text{ is continuous at } x = 1$$

ii) To show f is differentiable at $x = 1$

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} \Rightarrow \lim_{x \rightarrow 1^+} \frac{|x - 1|}{x - 1} = \lim_{x \rightarrow 1^+} \frac{x - 1}{x - 1} = 1$$

$$\text{and } \lim_{x \rightarrow 1^-} \frac{-x + 1}{x - 1} = \lim_{x \rightarrow 1^-} \frac{1 - x}{x - 1} = -1 \Rightarrow f'_+(1) = 1 \text{ and } f'_-(1) = -1$$

Hence f is not differentiable at $x = 1$.

We conclude that continuity does not imply differentiability.

b) Exercise left for you.

c) i) To show continuity at $x = 2$.

$$\lim_{x \rightarrow 2} f(x) = f(2) \Rightarrow \lim_{x \rightarrow 2} \sqrt[3]{x - 2} = f(2) = 0$$

 $\therefore f$ is continuous at $x = 2$.ii) To show f is differentiable at $x = 2$.

$$\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \text{ exist.}$$

$$\Rightarrow f'_+(2) = \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{2(2 - x)}{x - 2} = -2$$

$$\Rightarrow f'_-(2) = \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2^-} x + 2 = 4$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \text{ does not exist.}$$

 $\Rightarrow f$ is not differentiable but cont. at $x = 2$.

17. Let $f(x) = \begin{cases} 5x - 6 & \text{if } 0 < x < 1 \\ 4x^2 - 3x - 2 & \text{if } 1 \leq x \leq 2 \\ 3x + 2, & \text{if } x \geq 2 \end{cases}$ then which of the

following is not true about function.

- A. f is continuous at $x = 1$ C. f is differentiable at $x = 2$
 B. f is continuous at $x = 2$ D. f is differentiable at $x = 1$

Solution: A. $\lim_{x \rightarrow 1} f(x)$ exist and equal to $f(1)$.

$$\Rightarrow \lim_{x \rightarrow 1^+} (4x^2 - 3x - 2) = \lim_{x \rightarrow 1^-} 5x - 6 \Rightarrow -1 = -1$$

$\therefore f$ is cont. at $x = 1$.

$$\text{B. } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) \Rightarrow \lim_{x \rightarrow 2^+} 3x + 2 = \lim_{x \rightarrow 2^-} 4x^2 - 3x - 2$$

$$\Rightarrow 8 = 8 \Rightarrow f \text{ is cont. at } x = 2.$$

$$\text{C. } f'_+(2) = \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{3x + 2 - 8}{x - 2}$$

$$= \lim_{x \rightarrow 2^+} \frac{3(x - 2)}{x - 2} = 3$$

$$f'_-(2) = \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{4x^2 - 3x - 2 - 8}{x - 2}$$

$$= \lim_{x \rightarrow 2^-} \frac{(x - 2)(4x + 5)}{x - 2} = \lim_{x \rightarrow 2^-} 4x + 5 = 13$$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \text{ does not exist } (f'_+(2) \neq f'_-(2))$$

D. $f'(1)$ exist.

Answer: C

Derivative of some function

Basic differentiation rules

Differentiation of:

- power function
- simple trigonometric function
- exponential function
- logarithmic function

I. Derivative of constant function

The derivative of constant function is zero.

That is $f(x) = c \Rightarrow f'(x) = 0$ or $\frac{d}{dx}(c) = 0$.

Illustrative Example

18. Find the derivative of each of the following.

$$\text{a. } f(x) = \pi \quad \text{b. } f(x) = e^2 \quad \text{c. } f(x) = \log_4$$

Solution: Derivative of constant is zero.

$$\text{a. } f(x) = \pi \Rightarrow f'(x) = 0 \quad \text{c. } f(x) = \log_4 \Rightarrow f'(x) = 0$$

$$\text{b. } f(x) = e^2 \Rightarrow f'(x) = 0$$

Derivative rule of power functioni) If $f(x) = x^n$, then $f'(x) = nx^{n-1}$, $n \in \mathbb{R}$ ii) If $f(x) = kx^n$, then $f'(x) = knx^{n-1}$, k is constantiii) If $f(x) = \frac{k}{x^n}$ the same as $f(x) = kx^{-n}$, then $f'(x) = -knx^{-n-1}$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Illustrative Examples19. Find the derivative of each of the following functions with respect x .

a) $f(x) = x^{e-1}$ c) $f(x) = x^3 \sqrt{x^2}$ e) $f(x) = x^{\frac{2}{3}} \sqrt{x}$

b) $f(x) = x^{3-\pi}$ d) $f(x) = \frac{2}{x^3}$ f) $f(x) = \frac{1}{2\sqrt[3]{x^2}}$

Solution: a) $f(x) = x^{e-1} \Rightarrow f'(x) = (e-1)x^{e-2} = (e-1)x^{e-2}$

b) $f(x) = x^{3-\pi} \Rightarrow f'(x) = (3-\pi)x^{3-\pi-1} = (3-\pi)x^{2-\pi}$

c) $f(x) = x^3 \sqrt{x^2} \Leftrightarrow f(x) = x^1 \cdot x^{\frac{2}{3}} = x^{\frac{5}{3}} \Rightarrow f'(x) = \frac{5}{3}x^{\frac{2}{3}}$

d) $f(x) = \frac{2}{x^3} \Leftrightarrow f(x) = 2x^{-3} \Rightarrow f'(x) = -6x^{-4} = \frac{-6}{x^4}$

e) $f(x) = x^{\frac{2}{3}} \cdot x^{\frac{1}{2}} = x^{\frac{7}{6}} = \frac{7}{6}x^{\frac{1}{6}} = \frac{7}{6}\sqrt[6]{x}$

f) $f(x) = \frac{1}{2}x^{-\frac{2}{3}} \Rightarrow f'(x) = \left(\frac{1}{2}\right)\left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1}$
 $= \frac{-1}{3}x^{-\frac{5}{3}} = \frac{-1}{3\sqrt[3]{x^5}}$

g) $f(x) = x^{\sqrt{2}} \Rightarrow f'(x) = \sqrt{2}x^{\sqrt{2}-1}$

Let $f(x) = \sqrt[3]{x-2}$

a. Find $f'(x)$ 

$$\Rightarrow f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \text{ and } f'\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow y - f\left(\frac{\pi}{3}\right) = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) \Rightarrow y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$

$$\Rightarrow y = -\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}\pi}{6} + \frac{1}{2} \leftarrow \text{Equation of tangent line.}$$

2. Find x and y intercept of the tangent line to the graph of $f(x) = \sin x$, at $x = \frac{\pi}{3}$.

Solution: $y - f\left(\frac{\pi}{3}\right) = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) \leftarrow \text{Equation of tangent line.}$

$$f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \text{ and } f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\Rightarrow y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right) \Rightarrow y = \frac{1}{2}x - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

To find y intercept, set $x = 0$.

$$y = -\frac{\pi}{6} + \frac{\sqrt{3}}{2} \leftarrow y \text{ intercept}$$

To find x intercept set $y = 0$.

$$\Rightarrow \frac{1}{2}x - \frac{\pi}{6} + \frac{\sqrt{3}}{2} = 0 \Rightarrow x = \frac{\pi}{3} - \sqrt{3} \leftarrow x \text{ intercept}$$

IV.

i) **Derivative of exponential and logarithm function**

ii) If $f(x) = a^x$, then $f'(x) = a^x \ln a$

iii) If $f(x) = a^{mx}$, then $f'(x) = a^{mx} (m) \ln a$

iv) If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$

ii) If $f(x) = \log_a x$, then $f'(x) = \frac{1}{x} \log_a e = \frac{1}{x \ln a}$

Note: $\log_a^x = \frac{\log_c^x}{\log_c^a} = \frac{\ln x}{\ln a}$

To proof. Let $f(x) = a^x \Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h}$$

$$\therefore f'(x) = a^x \lim_{h \rightarrow 0} \left(\frac{a^h - 1}{h} \right) = a^x \ln a$$

$$\begin{aligned} y = a^x &\Rightarrow \frac{dy}{dx} = a^x \ln a \\ y = e^x &\Rightarrow \frac{dy}{dx} = e^x \ln e = e^x \end{aligned}$$

Illustrative Example

23. Find i) $f'(x)$

ii) Equation of tangent line to the graph of f at the indicated point.

a. $f(x) = 4^x$, at $(1, 4)$

e. $f(x) = (3^x)(2^{2x})$ at $(0, 1)$

b. $f(x) = \sqrt{6^x}$, at $(2, 6)$

f. $f(x) = \log_7^6$, at $(7, 1)$

c. $f(x) = (\pi + 2)^x$, at $(0, 1)$

g. $f(x) = \ln 4x$ at $\left(\frac{e}{4}, 1\right)$

d. $f(x) = e^{4x}$ at $(-1, e^{-4})$

Solution: a. i) $f(x) = 4^x \Rightarrow f'(x) = 4^x \ln 4$

ii) $y - f(1) = f'(1)(x - 1) \Rightarrow y - 4 = 4 \ln 4 (x - 1) \leftarrow$ Equation of tangent line.

b. i) $f(x) = \sqrt{6^x} \Rightarrow f'(x) = \sqrt{6^x} \ln \sqrt{6} = \frac{6^{\frac{1}{2}x}}{2} \ln 6$

ii) $y - f(0) = f'(2)(x - 2) \Rightarrow y - 6 = 3 \ln 6 (x - 2)$

c. i) $f(x) = (\pi + 2)^x \Rightarrow f'(x) = (\pi + 2)^x \ln (\pi + 2)$

ii) $y - f(0) = f'(0)(x - 0) \Rightarrow y - 1 = \ln (\pi + 2) \cdot x$

d. i) $f(x) = e^{4x} \Rightarrow f'(x) = e^{4x}(4) = 4e^{4x}$

$$\text{ii) } y - f(-1) = f'(-1)(x + 1) \Rightarrow y - e^{-4} = 4e^{-4}(x + 1)$$

$$\Rightarrow y = 4e^{-4}x + 5e^{-4} \leftarrow \text{Equation of tangent line.}$$

$$\text{e) i) } f(x) = (3^x)(2^{2x}) = (3^x)(4^x) = (12)^x \Rightarrow f'(x) = 12^x \ln(12)$$

$$\text{ii) } y - f(0) = f'(0)(x - 0) \Rightarrow y - 1 = (\ln(12))(x)$$

$$\therefore y = x \ln 12 + 1 \leftarrow \text{Equation of tangent line.}$$

$$\text{f) } f(x) = \log_7^x \Rightarrow f'(x) = \frac{1}{x \ln 7} = \frac{1}{x} \log_7^e$$

$$y - f(7) = f'(7)(x - 7) \Rightarrow y - 1 = \frac{1}{7} \log_7^e (x - 7) \leftarrow \text{Equation of tangent line.}$$

$$\text{g) i) } f(x) = \ln 4x \Rightarrow f'(x) = \frac{4}{4x} = \frac{1}{x}$$

$$\text{ii) } y - f\left(\frac{e}{4}\right) = f'\left(\frac{e}{4}\right)\left(x - \frac{e}{4}\right) \Rightarrow y - 1 = \frac{4}{e}\left(x - \frac{e}{4}\right) \Rightarrow y = \frac{4}{e}$$

$\leftarrow$ Equation of tangent.

Derivative of sum and difference

If f and g are differentiable at x_0 , then $f + g$ and $f - g$ are also differentiable at x_0 , so that:

- $(f + g)'(x_0) = f'(x_0) + g'(x_0) \leftarrow$ sum rule
- $(f - g)'(x_0) = f'(x_0) - g'(x_0) \leftarrow$ difference rule

Illustrative Example

24. Find a in each of the following cases.

a. $f(x) = x^3 - 5x$; if $f'(a) = 13$

b. $f(x) = \sin x + 6$ if $f'(a) = \frac{\sqrt{3}}{2}$ in $-\frac{\pi}{2} < a < \frac{\pi}{2}$.

Solution: a) $f'(x) = 3x^2 - 5 \Rightarrow f'(a) = 3a^2 - 5 = 13 \Rightarrow 3a^2 = 18 \Rightarrow a^2 = 6 \Rightarrow a = \pm\sqrt{6}$

b) $f'(x) = \cos x \Rightarrow f'(a) = \cos a = \frac{\sqrt{3}}{2} \Rightarrow a = \frac{\pi}{6}$

25. What is the equation of straight line having slope -4 that is tangent to the graph $y = 3x^2 + 2x + 1$. (UEE)

A. $y = 4x$ B. $y = 4x + 1$ C. $y = -4x - 2$ D. $y = -4x + 6$

Solution: Let $(a, f(a))$ point of tangency. Let $f(x) = 3x^2 + 2x + 1 \Rightarrow f'(x) = 6x + 2 \Rightarrow f'(a) = 6a + 2 = -4 \Rightarrow 6a = -6 \Rightarrow a = -1$

$$y - f(a) = f'(a)(x - a) \Rightarrow y - f(-1) = f'(-1)(x + 1)$$

$$\Rightarrow y - 2 = -4(x + 1)$$

$$\therefore y = -4x - 2 \leftarrow \text{Equation of tangent line.}$$

26. Let $f(x) = \cos x + 5^x$, then $f'(x) \dots$ UEE

$$\text{Solution: } \Rightarrow f'(x) = -\sin x + 5^x \ln 5$$

27. Find the value of k so that the line $y = -8x + k$ is tangent to the curve $y = 3x^2 + 4x + 1$.

Solution: Let $f(x) = 3x^2 + 4x + 1 \Rightarrow f'(x) = 6x + 4$. Let $(a, f(a))$ be point of tangency.

$$\Rightarrow f'(x) = 6x + 4 \Rightarrow f'(a) = 6a + 4 = -8 \leftarrow \text{slope of tangent line}$$

$$\Rightarrow 6a = -12 \Rightarrow a = -2, \text{ then eqn of tangent line } y - f(a) = f'(a)(x - a) \Rightarrow y - f(-2) = f'(-2)(x + 2) \Rightarrow y - 5 = -8(x + 2), \text{ since } f(-2) = 3(-2)^2 + 4(-2) + 1 = 5 \Rightarrow y = -8x - 16 + 5 = -8x - 11$$

28. Find the slope of the given curve at the indicated point.

$$\text{a. } f(x) = x^2 - 5x + 1; x = 1 \quad \text{b. } f(x) = x^2 + \frac{1}{x}, x = \sqrt{2}$$

$$\text{Solution: a) } f'(x) = 2x - 5 \Rightarrow f'(1) = 2 - 5 = -3 \leftarrow \text{slope}$$

$$\text{b) } f'(x) = 2x - \frac{1}{x^2} \Rightarrow f'(\sqrt{2}) = 2\sqrt{2} - \frac{1}{2} \leftarrow \text{slope}$$

29. Find the equation of all lines through the origin that are tangent to the curve $y = x^3 - 9x^2 - 16x$.

Solution: Let $f(x) = x^3 - 9x^2 - 16x$. Let $(a, f(a))$ is point of tangency pass through $(0, 0)$, so that slope m of tangent line is

$$\Rightarrow m = \frac{f(a) - 0}{a - 0} = \frac{f(a)}{a}, \text{ but slope of tangent line is obtained}$$

$$f'(x) = 3x^2 - 18x - 16 \Rightarrow f'(a) = 3a^2 - 18a - 16$$

$$\Leftrightarrow m = \frac{f(a)}{a} = f'(a) \Rightarrow \frac{a^3 - 9a^2 - 16a}{a} = 3a^2 - 18a - 16$$

$$\Rightarrow a^3 - 9a^2 - 16a = 3a^3 - 18a^2 - 16a \leftarrow \text{simplifying}$$

$$\Rightarrow 2a^3 - 9a^2 = 0 \Rightarrow a^2(2a - 9) = 0$$

$$\Rightarrow a^2 = 0 \text{ and } 2a - 9 = 0 \Rightarrow a = \frac{9}{2}$$

$$\therefore (a, f(a)) = \left(\frac{9}{2}, f\left(\frac{9}{2}\right) \right) = \left(\frac{9}{2}, -72 \right)$$

$\therefore$ Equation of tangent line through $(0, 0)$ and $\left(\frac{9}{2}, -72\right)$ is given

$$\text{by } \frac{y-0}{x-0} = \frac{0-(-72)}{0-\frac{9}{2}} = \frac{72}{-\frac{9}{2}} = -16$$

$$\therefore \frac{y}{x} = -16 \Rightarrow y = -16x \leftarrow \text{Equation of tangent line.}$$

8. Find the equation of the line tangent to the graph of $y = x^2 - 8x - 4$ at the point where the graph crosses $y = -3x + 2$.

Solution: The point where the graph crosses will be

$$\begin{cases} y = x^2 - 8x - 4 \\ y = -3x + 2 \end{cases} \Rightarrow x^2 - 8x - 4 = -3x + 2$$

$\Rightarrow x^2 - 5x - 6 = 0 \Leftrightarrow (x-6)(x+1) = 0 \Rightarrow x = 6$ and $x = -1$, so that when $x = -1$, $y = 5 \Rightarrow (-1, 5)$. The equation of tangent line will be:

$$\Rightarrow y - f(-1) = f'(-1)(x + 1) \Rightarrow y - 5 = -10(x + 0) \leftarrow f'(x) = 2x - 8$$

$$\Rightarrow f'(-1) = -10 \Rightarrow y = -10x - 5 \leftarrow \text{Equation of tangent line.}$$

Derivatives of product and quotient of function

A. Product Rule

If f and g are differentiable at x then the product of f and g is differentiable at x and its derivative is given by:

$$\text{i) } (fg)'(x) = f'(x)g(x) + f(x)g'(x) \leftarrow \text{product rule}$$

$$\text{ii) } (fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

B. Quotient Rule

$$\text{i) } \left(\frac{f}{g} \right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2}$$

Note: If $f(x) = 1 \Rightarrow f'(x) = 0 \Rightarrow \left(\frac{1}{g(x)} \right)' = \frac{0 - g'(x)(1)}{[g(x)]^2} = \frac{-g'(x)}{[g(x)]^2}$

$$\bullet \text{ If } y = \left(\frac{f}{g} \right)(x) \text{ then } \frac{dy}{dx} = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}.$$

Illustrative Example

31. Given that $g(8) = -3$, $g'(8) = 5$, $f(8) = -1$, and $f'(8) = 4$. Then find $F'(8)$ if:

a. $F(x) = \sqrt[3]{x} f(x)$

d. $F(x) = f(x) g(x)$

b. $F(x) = \frac{3x-1}{g(x)}$

e. $F(x) = \frac{x+f(x)}{g(x)}$

c. $F(x) = \frac{f(x)}{g(x)}$

f. $F(x) = \frac{g(x)}{x-3}$

Solution: Here, using the rule

a. $F'(x) = \frac{1}{3} x^{-\frac{2}{3}} f(x) + \sqrt[3]{x} f'(x) \leftarrow \text{Product rule}$

$$\Rightarrow F'(8) = \frac{1}{3} (8)^{-\frac{2}{3}} (8) + \sqrt[3]{8} f'(8) = -\frac{1}{12} + 2(4) = \frac{95}{12}$$

b. $F'(x) = \frac{3(g(x)) - g'(x)(3x-1)}{[g(x)]^2} \leftarrow \text{quotient rule}$

$$\Rightarrow F'(8) = \frac{3g(8) - g'(8)(23)}{[g(8)]^2} = \frac{-9 - 5(23)}{[-3]^2} = \frac{-124}{9}$$

c. $F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \leftarrow \text{quotient rule}$

$$\Rightarrow F'(8) = \frac{f'(8)g(8) - f(8)g'(8)}{[g(8)]^2} = \frac{4(-3) - (-1)(5)}{(-3)^2} = \frac{-7}{9}$$

d. $F'(x) = f'(x)g(x) + f(x)g'(x) \leftarrow \text{product rule}$

$$\Rightarrow F'(8) = f'(8)g(8) + f(8)g'(8) = 4(-3) + (-1)(5) = -17$$

e. $F'(x) = \frac{(1+f'(x))g'(x)(x+f(x))}{[g(x)]^2} \leftarrow \text{quotient rule}$

$$\Rightarrow F'(8) = \frac{(1+f'(8))g'(8)(8+f(8))}{[g(8)]^2}$$

$$= \frac{(1+4)(-3) - 5(7)}{[-3]^2} = \frac{-50}{9}$$

f. Exercise left for you.

Find an equation of the line tangent to the graph of f at the point $(a, f(a))$.

a. $f(x) = \frac{x+1}{x-1}$, at $a = 3$ d. $f(x) = \frac{\ln x}{x}$, $a = e$

b. $f(x) = xe^x$, at $a = 1$ e. $f(x) = e^x (\sin x + \cos x)$, $a = 0$

c. $f(x) = \frac{2x^2-1}{x^2+1}$; $a = \sqrt{3}$

Solution: Here

a. $f'(x) = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$ ← quotient rule

$$y - f(3) = f'(3)(x - 3) \Rightarrow y - 2 = -\frac{1}{2}(x - 3)$$

$$\therefore f(3) = 2 \Leftrightarrow 2y + x = 7 \leftarrow \text{Equation of tangent line}$$

b. $f(x) = xe^x \Rightarrow f'(x) = (1)e^x + xe^x = e^x(x+1)$ ← product rule
 $\Rightarrow f(1) = e$ and $f'(1) = 2e$.

$$\Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - e = 2e(x - 1) \Rightarrow y = 2ex - e \leftarrow \text{Equation of tangent line}$$

c. $f(x) = \frac{2x^2-1}{x^2+1} \Rightarrow f'(x) = \frac{4x(x^2+1) - (2x^2-1)(2x)}{(x^2+1)^2}$

$$f'(x) = \frac{6x}{(x^2+1)^2} \leftarrow \text{simplifying}$$

$$\Rightarrow f(\sqrt{3}) = \frac{2(3)-1}{3+1} = \frac{5}{4} \text{ and } f'(\sqrt{3}) = \frac{6\sqrt{3}}{16} = \frac{3\sqrt{3}}{8}$$

$$\Rightarrow y - f(\sqrt{3}) = f'(\sqrt{3})(x - \sqrt{3}) \Rightarrow y - \frac{5}{4} = \frac{3\sqrt{3}}{8}(x - \sqrt{3})$$

$$\Rightarrow y = \frac{3\sqrt{3}x}{8} - \frac{9}{8} + \frac{5}{4} \Rightarrow y = \frac{3\sqrt{3}x}{8} + \frac{1}{8} \leftarrow \text{Equation of tangent line}$$

$$d. \quad f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{\frac{1}{x}(x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$$\Rightarrow f(e) = \frac{1}{e} \text{ and } f'(e) = 0. \text{ Therefore the equation of tangent}$$

$$\text{line is } y - f(e) = f'(e)(x - e) \Rightarrow y - \frac{1}{e} = 0 \Rightarrow y = \frac{1}{e}$$

$$e. \quad f(x) = e^x \sin x + e^x \cos x$$

$$\Rightarrow f'(x) = e^x \cos x + e^x \sin x + e^x \cos x + e^x (-\sin x)$$

$$\Rightarrow f'(x) = 2e^x \cos x \leftarrow \text{simplifying}$$

$$\text{Then, } f(0) = 1, f'(0) = 2, \text{ the equation of tangent line is}$$

$$-f(0) = f'(0)(x - 0) \Rightarrow y - 1 = 2x \Rightarrow y = 2x + 1$$

33. The equation of tangent line to the graph of $f(x) = \frac{3x}{x-1}$ at

$$(2, f(2)). \text{ UEE}$$

A. $3y - x = 20$

C. $y - 3x = 0$

B. $3y - x = 16$

D. $y + 3x = 12$

Solution: $f'(x) = \frac{3(x-1) - 3x(1)}{(x-1)^2} = \frac{-3}{(x-1)^2} \leftarrow \text{quotient rule}$

$$\Rightarrow f(2) = \frac{6}{2-1} = 6 \text{ and } f'(2) = -3 \Rightarrow y - f(2) = f'(2)(x - 2) \Rightarrow y$$

$$- 6 = -3(x - 2) \Rightarrow y + 3x = 12$$

$$\therefore y + 3x = 12 \leftarrow \text{Eqn of tangent line}$$

Answer: D

34. Let $f(x) = \frac{1}{3^x + 1}$, find a) $f'(x)$

b) Equation of tangent line to the graph of f at $x = 0$.

Solution: Recall $\left(\frac{1}{g(x)} \right)' = \frac{-g'(x)}{[g(x)]^2}$

a. $f'(x) = \frac{-(3^x + 1)'}{(3^x + 1)^2} = \frac{-3^x \ln 3}{(3^x + 1)^2}$

b. $f'(0) = \frac{-(3^0 \ln 3)}{4} = \frac{-\ln 3}{4} \leftarrow \text{slope}$

$$f(0) = f'(0)(x-0) \Rightarrow y - \frac{1}{2} = \left(\frac{-\ln 3}{4} \right) x \Rightarrow y = \left(\frac{-\ln 3}{4} \right) x + \frac{1}{2}$$

Let $f(x) = x^2 e^x \sin x$, then find $f'(x)$.

Solution: use $(fgh)' = f'gh + fg'h + fgh'$

$$\Rightarrow f'(x) = 2xe^x \sin x + x^2 (-e^x) \sin x + x^2 e^x \cos x$$

$$= xe^x (2 \sin x - x \sin x + x \cos x)$$

Find the derivative of the following.

a. $f(x) = \tan x$

d. $f(x) = \frac{1}{\ln x}$

b. $f(x) = \csc x$

e. $f(x) = \frac{1}{x^2 - 4x + 6}$

c. $f(x) = \sec x$

f. $f(x) = \frac{1}{(x-2)^2}$

Solution: Here, using rule of derivative.

$$f(x) = \tan x \Rightarrow \frac{\sin x}{\cos x} \Rightarrow f'(x) = \frac{(\cos x) \cos x - (-\sin x) \sin x}{\cos^2 x}$$

$$\Rightarrow \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Explanation: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

$$(\tan x)' = \sec^2 x$$

$$f(x) = \csc x \Leftrightarrow f(x) = \frac{1}{\sin x} \leftarrow \text{use } \left(\frac{1}{g(x)} \right)' = \frac{-g'(x)}{(g(x))^2}$$

$$(\csc x)' = -\cot x \csc x$$

$$f(x) = \sec x \Leftrightarrow f(x) = \frac{1}{\cos x}, \text{ use } \left[\frac{1}{g(x)} \right]' = \frac{g'(x)}{[g(x)]^2}$$

$$\Rightarrow f'(x) = \frac{-\sin x}{(\cos x)^2} = \frac{-\sin x}{\cos x} \cdot \frac{1}{\cos x} = -\tan x \sec x$$

$$d. \quad f(x) = \frac{1}{\ln x} \Rightarrow f'(x) = \frac{-\frac{1}{x}}{(\ln x)^2} = \frac{-1}{x(\ln x)^2}$$

$$e. \quad f(x) = \frac{1}{x^2 - 4x + 6} \Rightarrow f'(x) = \frac{-2x + 4}{x^2 - 4x + 6}$$

$$f. \quad f(x) = \frac{1}{(x-2)^2} \Rightarrow f'(x) = \frac{-2(x-2)}{(x-2)^4} = \frac{-2}{(x-2)^3}$$

37. In each of the following function, find $\frac{dy}{dx}$.

$$a. \quad y = \frac{x \sin x}{x - e^{-x}}$$

$$d. \quad y = \frac{x^2 + 4}{\cos x}$$

$$b. \quad y = \frac{\sin x}{1 - \cos x}$$

$$e. \quad y = \frac{\sqrt{x}}{x \sin x}$$

$$c. \quad y = \frac{e^x \sin x}{e^x + 1}$$

$$f. \quad y = e^x (x^2 - 1) \tan x$$

Solution: If $y = f(x) \Rightarrow$ then $\frac{dy}{dx} = f'(x)$

$$a. \quad y = \frac{x \sin x}{x - e^{-x}} \leftarrow \frac{dy}{dx} \text{ (exercise left for you)}$$

$$b. \quad y = \frac{\sin x}{1 - \cos x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{\sin x}{1 - \cos x} \right)$$

$$= \frac{(1 - \cos x) \frac{d}{dx} (\sin x) - \sin x \frac{d}{dx} (1 - \cos x)}{(1 - \cos x)^2}$$

$$= \frac{(1 - \cos x)(\cos x) - \sin x \sin x}{(1 - \cos x)^2} = \frac{\cos x - \cos^2 x - \sin^2 x}{(1 - \cos x)^2}$$

$$= \frac{\cos x - (\cos^2 x + \sin^2 x)}{(1 - \cos x)^2} = \frac{\cos x - 1}{(1 - \cos x)^2} = \frac{-1}{1 - \cos x}$$

$$\begin{aligned}
 y &= \frac{x^2 + 4}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left[\frac{x^2 + 4}{\cos x} \right] \\
 &\Rightarrow \frac{\cos x \frac{d}{dx} (x^2 + 4) - (x^2 + 4) \frac{d}{dx} (\cos x)}{(\cos x)^2} \\
 &\Rightarrow \frac{2x \cos x + (x^2 + 4) \sin x}{\cos^2 x} = 2x \sec x + (x^2 + 4) \tan x \sec x
 \end{aligned}$$

Conceptual Example

Q. If $f(x)$ is differentiable at $x = a$, then what is the value of

$$\lim_{x \rightarrow a} \frac{x f(a) - a f(x)}{x - a}$$

Explanation:

Add $-af(a) + af(a)$ to the expression

$$\lim_{x \rightarrow a} \frac{x f(a) - a f(x)}{x - a} = \lim_{x \rightarrow a} \frac{x f(a) - a f(a) + a f(a) - a f(x)}{x - a}$$

$$\lim_{x \rightarrow a} f(a) \frac{(x - a)}{x - a} + \lim_{x \rightarrow a} a \frac{f(a) - f(x)}{x - a}$$

$$= f(a) - a \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f(a) - a f'(a)$$

For what value of a and b is the function.

$$f(x) = \begin{cases} 1 - 4x^2, & \text{for } x \leq 1 \\ ax + b, & \text{for } x > 1 \end{cases} \text{ differentiable at } x = 1$$

Explanation:

$$f(x) = 1 - 4x^2 \Rightarrow f'(x) = -8x, \text{ for } x < 1$$

$$f(x) = ax + b \Rightarrow f'(x) = a, \text{ for } x > 1$$

Since $f(x)$ is differentiable at $x = 1$.

$$f'_+(1) = a = -8 = f'_-(1)$$

Also differentiability implies continuity

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1) \Rightarrow \lim_{x \rightarrow 1} ax + b = 1 - 4 = -3$$

$$\Rightarrow a + b = -3, \text{ but } a = -8 \Rightarrow b = 5$$

The Chain Rule

One of the most powerful differentiation rules the chain rule. This rule deals with composite function (derivative of function of function).

Theorem (the chain rule): If g is differentiable at x and f is differentiable at $g(x)$ then $f(g(x))$ is differentiable at x . So that $(f(g(x)))' = f'(g(x)) g'(x)$.

Equivalently, let $y = f(u)$ and $u = g(x)$.

$$\text{Then } y = f(g(x)) \Rightarrow \frac{dy}{du} = f'(u) \text{ and } \frac{du}{dx} = g'(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(f(g(x))) = f'(g(x)) g'(x) = f'(u) \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \leftarrow \text{other notation of chain rule}$$

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a) $y = \frac{1}{x+4}$	$u = x+4$	$y = \frac{1}{u}$
b) $y = \cos(x^2+4)$	$u = x^2+4$	$y = \cos u$
c) $y = \sin^3 x$	$u = \sin x$	$y = u^3$
d) $y = \sqrt{x^6+1}$	$u = x^6+1$	$y = \sqrt{u}$
e) $y = \ln(x^2+4x)$	$u = x^2+4x$	$y = \ln u$
f) $y = e^{x^2-1}$	$u = x^2-1$	$y = e^u$
g) $y = a^{f(x)}$	$u = f(x)$	$y = a^u$

The General rule

1. If $y = [f(x)]^n$, then $\frac{dy}{dx} = n(f(x))^{n-1} f'(x) \leftarrow$ General power rule.
2. If $y = \sin(f(x))$, then $\frac{dy}{dx} = \cos(f(x)) f'(x) \leftarrow$ chain rule to

Trigonometric function

3. Let $y = \cos^n(f(x))$, means $y = [\cos(f(x))]^n$ then

$\frac{dy}{dx} = -n [\cos (f(x))]^{n-1} \sin f(x) \cdot f'(x) \leftarrow$ General trigonometric function rule

Illustrative Example

Find the derivative of the following function using chain rule.

- a. $y = (x^2 + 1)^3$ b. $y = (3 - 5x)^4$ c. $y = \sqrt{x^2 - 4x + 5}$
 d. $y = \cos (x^2 + 6)$ e. $y = \sin^5 (3x^2 + 1)$ f. $y = \sin^3 4x$
 g. $y = \log_{10} (x^2 + 4)$ h. $y = 4^{x^2 + 3x}$ i. $y = \ln \sqrt{x^2 + 1}$
 j. $y = \sqrt{\log \sqrt{x}}$ k. $y = \log_{10} \sqrt{x^2 + 4}$

Solution: a. $y = (x^2 + 1)^3$. Let $u = x^2 + 1 \Rightarrow y = u^3 \Rightarrow \frac{du}{dx} = 2x$

$$\text{and } \frac{dy}{du} = 3u^2 \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 (2x) = 6x(x^2 + 1)^3.$$

In short: use $y = [f(x)]^n \Rightarrow \frac{dy}{dx} = n[f(x)]^{n-1} f'(x)$

Let $f(x) = x^2 + 1 \Rightarrow y = [f(x)]^3$

$$\Rightarrow \frac{dy}{dx} = 3[f(x)]^2 f'(x) = 3[x^2 + 1]^2 (2x) = 6x[x^2 + 1]^2$$

b. Let $f(x) = 3 - 5x \Rightarrow y = [f(x)]^4$ and $f'(x) = -5$

$$\Rightarrow \frac{dy}{dx} = 4[f(x)]^3 f'(x) = 4[3 - 5x]^3 (-5) = -20[3 - 5x]^3$$

c. Let $f(x) = x^2 - 4x + 5 \Rightarrow y = [f(x)]^{\frac{1}{2}}$ and $f'(x) = 2x - 4$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}[f(x)]^{-\frac{1}{2}} f'(x) = \frac{f'(x)}{2\sqrt{f(x)}} = \frac{2x - 4}{2\sqrt{x^2 - 4x + 5}}$$

d. Let $f(x) = x^2 + 6 \Rightarrow y = \cos f(x)$ and $f'(x) = 2x$

$$\Rightarrow \frac{dy}{dx} = -\sin (f(x)) \quad f'(x) = -\sin (x^2 + 6) (2x) = -2x \sin (x^2 + 6)$$

In short, $\frac{dy}{dx} = \frac{d}{dx} (\cos(x^2 + 6)) = -\sin(x^2 + 6) \frac{d}{dx} (x^2 + 6) = -2x \sin (x^2 + 6)$

$$e. y = \sin^5(3x^2 + 1) \Rightarrow y = [\sin(x^2 + 1)]^5$$

$$\frac{dy}{dx} = 5[\sin(x^2 + 1)]^4 \cos(x^2 + 1)(2x)$$

$$= 10x[\sin(x^2 + 1)]^4 \cos(x^2 + 1)$$

$$f. y = \sin^3 4x \Rightarrow y = (\sin 4x)^3$$

$$\Rightarrow \frac{dy}{dx} = 3(\sin 4x)^2 \cos 4x (4) = 12 \cos 4x \sin^2 4x$$

$$g. y = \log_{10}^{(x^2+4)} \text{ let } u = x^2 + 4 \Rightarrow \frac{du}{dx} = 2x$$

$$\Rightarrow y = \log_{10}^u \Rightarrow \frac{dy}{du} = \frac{1}{u} \log_{10}^e = \frac{1}{u \ln 10}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u \ln 10} \cdot (2x) = \frac{2x}{(x^2 + 4) \ln 10}$$

$$h. \text{ Use } y = a^{f(x)} \Rightarrow \frac{dy}{dx} = a^{f(x)} \ln(a) f'(x)$$

$$\text{Let } f(x) = x^2 + 3x \Rightarrow y = 4^{f(x)} \Rightarrow \frac{dy}{dx} = 4^{f(x)} \ln 4 (f'(x))$$

$$\Rightarrow \frac{dy}{dx} = 4^{x^2+3x} \ln(4)(2x+3)$$

$$i. y = \ln \sqrt{x^2 + 1} \Rightarrow \frac{dy}{dx} = \frac{f'(x)}{f(x)} = \frac{\frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}}{\sqrt{x^2 + 1}} = \frac{x}{x^2 + 1}$$

$$j. y = \sqrt{\log \sqrt{x}} \text{ let } u = \log_{10}^{\sqrt{x}} = \log_{10}^{\frac{1}{2}} = \frac{1}{2} \log_{10}^x$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2x \ln(10)} \text{ and } y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{\log_{10}^x}}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{\log_{10}^x} \cdot (2x) \ln(10)} = \frac{1}{4x\sqrt{\log_{10}^x} \ln 10}$$

$$k. y = \log_{10}^{\sqrt{x^2+4}} \text{ let } u = \sqrt{x^2 + 4} \Rightarrow u^2 = x^2 + 4$$

$$\Rightarrow \frac{d}{dx} u^2 = \frac{d}{dx} (x^2 + 4) = 2u \frac{du}{dx} = 2x \Rightarrow \frac{du}{dx} = \frac{x}{u} = \frac{x}{\sqrt{x^2 + 4}} \text{ so that}$$

$$\Rightarrow y = \log_{10} \sqrt{x^2 + 4} = \log_{10} u \Rightarrow \frac{dy}{du} = \frac{1}{u \ln 10}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u \ln 10} \cdot \frac{x}{\sqrt{x^2 + 4}} = \frac{x}{(x^2 + 4) \ln 10}$$

38. For each of the following function, find the points where the tangent line is parallel to the x -axis.

a. $y = x^2 + 4x - 7$

b. $y = \frac{x}{x^2 + 4}$

c. $y = \frac{x+3}{x-1}$

Solution: a) $\frac{dy}{dx} = 2x + 4$

Parallel to the x -axis means slope equal to zero, that is $2x+4=0 \Rightarrow x=-2$ so that $(-2, f(-2)) = (-2, -11) \Rightarrow y = -11 \leftarrow$ tangent line parallel to x -axis.

b) $y = \frac{x}{x^2 + 4} \Rightarrow \frac{dy}{dx} = \frac{(x^2 + 4) \frac{d}{dx} x - x \frac{d}{dx} (x^2 + 4)}{(x^2 + 4)^2} = \frac{4 - x^2}{(x^2 + 4)^2}$

$$\Rightarrow \text{set: } \frac{dy}{dx} = 0 \Rightarrow \frac{4 - x^2}{(x^2 + 4)^2} = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2$$

$$\Rightarrow (2, f(2)) = \left(2, \frac{1}{4}\right) \text{ and } \left(-2, -\frac{1}{4}\right)$$

c) Exercise left for you.

40. Find the point on the curve $y = x^3 + x^2 + x$ where the tangent line is parallel to $y = 2x + 3$.

Solution: $\frac{dy}{dx} = 3x^2 + 2x + 1 = 2 \rightarrow$ slope of $y = 2x + 3$

$$\Rightarrow 3x^2 + 2x - 1 = 0 \Leftrightarrow (3x - 1)(x + 1) = 0 \Rightarrow 3x - 1 = 0$$

$$\Rightarrow 3x = 1 \Rightarrow x = \frac{1}{3} \text{ and } x = -1. \quad \therefore (-1, 1) \text{ and } \left(\frac{1}{3}, \frac{17}{3}\right)$$

$$\Rightarrow y - f(-1) = f'(-1)(x + 1) \Rightarrow y - 1 = 2x + 2 \Rightarrow y = 2x + 3$$

41. Find $\frac{dy}{dx}$ if:

a. $y = (4x^3 - x^2)^5$

e. $y = \ln \sqrt{\sin(x^2 + 1)}$

b. $y = \frac{1}{(2-5x)^3}$

f. $y = \sin(x^2) - \cos(x^2)$

c. $y = \tan(x - \sin 2x)$

g. $y = \sqrt{\frac{x}{x^2 + 1}}$

d. $y = \left(\frac{2x-1}{3-4x}\right)^7$

h. $y = \frac{3x^2}{\sqrt{1+x^3}}$

Solution:

a. $\frac{dy}{dx} = \frac{d}{dx} (4x^3 - x^2)^5 = 5(4x^3 - x^2)^4 \frac{d}{dx} (4x^3 - x^2)$
 $= 5(4x^3 - x^2)^4 (12x^2 - 2x)$

b. $\frac{dy}{dx} = \frac{d}{dx} (2-5x)^{-3} = -3(2-5x)^{-4} \frac{d}{dx} (2-5x)$
 $= -3(2-5x)^{-4} (-5) = \frac{15}{(2-5x)^4}$

c. $\frac{dy}{dx} = \frac{d}{dx} (\tan(x - \sin 2x))$
 $= \sec^2(x - \sin 2x) \frac{d}{dx} (x - \sin x) \frac{d}{dx} (2x)$
 $= \sec^2(x - \sin 2x) (1 - \cos 2x) (2)$
 $= 2 \sec^2(x - \sin 2x) (1 - \cos 2x)$

d. $\frac{dy}{dx} = \frac{d}{dx} \left(\frac{2x-1}{3-4x}\right)^7 = 7 \left(\frac{2x-1}{3-4x}\right)^6 \frac{d}{dx} \left(\frac{2x-1}{3-4x}\right)$
 $= 7 \left(\frac{2x-1}{3-4x}\right)^6 \frac{(3-4x) \frac{d}{dx} (2x-1) - (2x-1) \frac{d}{dx} (3-4x)}{(3-4x)^2}$

$$= 7 \left(\frac{2x-1}{3-4x} \right)^6 \frac{(3-4x)(2) - (2x-1)(-4)}{(3-4x)^2} = \frac{70(2x-1)^6}{(3-4x)^8}$$

$$= \frac{dy}{dx} = \frac{d}{dx} \ln \sqrt{\sin(x^2+1)} = \frac{d}{dx} \left(\frac{1}{2} \ln(\sin(x^2+1)) \right)$$

$$= \frac{\frac{d}{dx}(\sin(x^2+1)) \cdot \frac{d}{dx}(x^2+1)}{2 \sin(x^2+1)} = \frac{\cos(x^2+1)(2x)}{2 \sin(x^2+1)} = x \cot(x^2+1)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} (\sin(x^2) + \cos(x^2)) = \cos x^2 \frac{d}{dx} x^2 + \sin x^2 \frac{d}{dx} (x^2) \\ &= 2x \cos(x^2) - 2x (\sin x^2) = 2x (\cos(x^2) - \sin(x^2)) \end{aligned}$$

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sqrt{\frac{x}{x^2+1}} \right) = \frac{d}{dx} \left(\frac{x}{x^2+1} \right)^{\frac{1}{2}} = \frac{1}{2} \left(\frac{x}{x^2+1} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{x}{x^2+1} \right)$$

$$= \frac{\frac{1}{2} \sqrt{x^2+1} (x^2+1) \frac{d}{dx}(x) - x \frac{d}{dx}(x^2+1)}{(x^2+1)^2} \leftarrow \text{quotient rule}$$

$$= \frac{1-x^2}{2\sqrt{(x^2+1)^3}} \leftarrow \text{simplifying}$$

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{3x^2}{\sqrt{1+x^3}} \right) = \frac{\sqrt{1+x^3} \frac{d}{dx}(3x^2) - 3x^2 \frac{d}{dx} \sqrt{1+x^3} \cdot \frac{d}{dx}(1+x^3)}{(\sqrt{1+x^3})^2}$$

$$= \frac{6x\sqrt{1+x^3} - \frac{9x^4}{\sqrt{1+x^3}}}{1+x^3} = \frac{12x + 3x^4}{2\sqrt{(1+x^3)^3}}$$

42. Find the equation of the line tangent to the graph of f at $(a, f(a))$.

a. $f(x) = \ln\left(\frac{x^3}{x^3 + x}\right)$ at $(1, f(1))$ e. $f(x) = (1 - x^3)\sqrt{2 - x}$ at $(-2, 18)$

b. $f(x) = e^{4-x^2}$ at $(1, e^3)$ f. $f(x) = \sin(\ln \sqrt{e^x})$ at $(0, 0)$

c. $f(x) = \ln\left(\frac{x+2}{\sqrt{x+3}}\right)$ at $(1, f(1))$ g. $f(x) = \ln\left(\frac{\cos x}{x+1}\right)$ at $(0, 0)$

d. $f(x) = \frac{e^{4x+1}}{3-2x}$ at $(1, e^5)$

Solution: Here,

Remember that: $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

a. $f(x) = \ln(x^3) - \ln(x^3 + x) \Rightarrow f'(x) = \frac{3}{x} - \frac{3x+1}{x^3+x}$

$f(1) = 0 - \ln(2)$ and $f'(1) = 3 - \frac{4}{2} = 1$

$y - f(1) = f'(1)(x - 1) \Rightarrow y + \ln 2 = 1(x - 1) \Leftrightarrow y = x - 1 - \ln 2$

b. $f'(x) = \frac{d}{dx}(e^{4-x^2}) = e^{4-x^2} \frac{d}{dx}(4-x^2) = e^{4-x^2}(-2x)$

$\Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - e^3 = -2e^3(x - 1) \Rightarrow f'(1) = -2e^3$

$\Rightarrow y = 2e^3x + 4e^3 \leftarrow$ Equation of tangent line

c. $f(x) = \ln\left(\frac{x+2}{\sqrt{x+3}}\right) = \ln(x+2) - \frac{1}{2}\ln(x+3)$

$\Rightarrow f'(x) = \frac{1}{x+2} - \frac{1}{2(x+3)}$

$\Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - \ln\left(\frac{3}{2}\right) = \frac{5}{24}(x - 1)$

$\Rightarrow y = \frac{5}{24}x - \frac{5}{24} + \ln\left(\frac{3}{2}\right) \leftarrow$ Equation of tangent

$$d. f'(x) = \frac{d}{dx} \left(\frac{e^{4x+1}}{3-2x} \right) = \frac{(3-2x) \frac{d}{dx} e^{4x+1} - e^{4x+1} \frac{d}{dx} (3-2x)}{(3-2x)^2}$$

$$= \frac{(3-2x)(4)e^{4x+1} - e^{4x+1}(-2)}{(3-2x)^2} = \frac{e^{4x+1}(14-8x)}{(3-2x)^2}$$

$$\Rightarrow f'(1) = \frac{e^5(14-8)}{1} = 6e^5 \leftarrow \text{slope}$$

$$y - f(1) = f'(1)(x-1) \Rightarrow y - e^5 = 6e^5(x-1) \Rightarrow y = 6e^5x - 5e^5$$

$$e. f'(x) = \frac{d}{dx} (1-x^3) \left[\sqrt{2-x} \right] + (1-x^3) \frac{d}{dx} \sqrt{2-x} \cdot \frac{d}{dx} (2-x)$$

$$= -3x^2 \left(\sqrt{2-x} + \frac{(1-x^3)}{2\sqrt{2-x}}(-1) \right) \text{ so that } f'(-2) = -\frac{105}{4}$$

$$y - f(-2) = f'(-2)(x+2) \Rightarrow y - 18 = -\frac{105}{4}(x+2) \leftarrow \text{tangent line}$$

$$f. f(x) = \sin \left(\ln \sqrt{e^x} \right) = \sin \left(\ln e^{\frac{1}{2}x} \right) = \sin \left(\frac{1}{2}x \ln e \right)$$

$$f(x) = \sin \left(\ln \sqrt{e^x} \right) = \sin \left(\ln e^{\frac{1}{2}x} \right) = \sin \left(\frac{1}{2}x(\ln e) \right) = \sin \left(\frac{1}{2}x \right)$$

$$\Rightarrow f'(x) = \cos \frac{1}{2}x \cdot \frac{1}{2} = \frac{1}{2} \cos \frac{1}{2}x \Rightarrow f'(0) = \frac{1}{2} \cos(0) = \frac{1}{2}(1) = \frac{1}{2}$$

$$y - f(0) = f'(0)(x-0) = y - 0 = \frac{1}{2}(x) \Rightarrow y = \frac{1}{2}x \leftarrow \text{tangent line}$$

$$g. f(x) = \ln \left(\frac{\cos x}{x+1} \right) = \ln(\cos x) - \ln(x+1)$$

$$\Rightarrow f'(x) = \frac{\sin x}{\cos x} - \frac{1}{x+1}, \text{ thus } f'(0) = 0 - 1 = -1$$

$$y - f(0) = f'(0)(x-0) \Rightarrow y - 0 = -1(x) = y = -x \leftarrow \text{tangent line}$$

43. Let $f(x) = \sec^3\left(\frac{\pi}{2} - x\right)$. Find: a) $f'(x)$ b) $f'\left(-\frac{\pi}{2}\right)$

Solution: $f'(x) = 3 \sec^2\left(\frac{\pi}{2} - x\right) \frac{d}{dx} \left(\sec\left(\frac{\pi}{2} - x\right) \frac{d}{dx} \left(\frac{\pi}{2} - x \right) \right)$

$$\Rightarrow f'(x) = 3 \sec^2\left(\frac{\pi}{2} - x\right) \sec\left(\frac{\pi}{2} - x\right) \tan\left(\frac{\pi}{2} - x\right) (-1)$$

b) $f'\left(-\frac{\pi}{2}\right) = 3 \sec^2(\pi) \sec(\pi) \tan(\pi) (-1) = 3(1)(-1)(0) = 0$

44. Find the slope of the line tangent to the curve $y = \sin^5 x$ at $x = \frac{\pi}{3}$.

Solution: $\frac{dy}{dx} = 5 \sin^4 x \frac{d}{dx} \sin x = 5 \sin^4 x \cos x$

The tangent line has slope $\left. \frac{dy}{dx} \right|_{x=\frac{\pi}{3}} = 5 \left(\frac{\sqrt{3}}{2} \right)^4 \left(\frac{1}{2} \right) = \frac{45}{32}$

45. Given that $f(2) = 1$, $f(8) = 5$, $f'(2) = 7$ and $f'(8) = -3$.

a. If $g(x) = [f(x)]^3$, then find $g'(2)$.

b. If $h(x) = f(x^3)$, then find $h'(2)$.

Solution: a) $g(x) = [f(x)]^3 \Rightarrow g'(x) = 3(f(x))^2 f'(x)$
 $\Rightarrow g'(2) = 3(f(2))^2 \cdot f'(2) = 3(1)^2(7) = 21$

b) $h(x) = f(x^3) \Rightarrow h'(x) = f'(x^3) \cdot 3x^2$

$$\Rightarrow h'(2) = f'(2^3) \cdot 3(2)^2 = (-3)(12) = -36$$

46. If $f(-2) = 3$, $g(2) = -2$, $g'(2) = -1$ and $f'(-2) = 5$ then find

a) $\left. \frac{d}{dx} f(g(x)) \right|_{x=2}$

b) $\left. \frac{d}{dx} \sqrt{f(x)} \right|_{x=2}$

Solution: Here,

a) $\left. \frac{d}{dx} (f(g(x))) \right|_{x=2} = \left. \frac{d}{dx} f(g(x)) \frac{d}{dx} g(x) \right|_{x=2} = f'(g(x)) g'(x) \Big|_{x=2} = -5$

b) $\left. \frac{d}{dx} \sqrt{f(x)} \right|_{x=2} = \frac{\frac{d}{dx} f(x)}{2\sqrt{f(x)}} = \frac{f'(x)}{2\sqrt{f(x)}} = \frac{f'(-2)}{2\sqrt{f(-2)}} = \frac{5}{2\sqrt{3}}$

47. If $f(2) = -3$, $f'(2) = 4$, $g(1) = 1$, $g'(1) = 1$, $g'(1) = 5$ and $F(x) = f(2x+2) \cdot g(1-x^2)$, then $F'(0) \dots$ UEE
 A. 4 B. 8 C. -11 D. -22

Solution: $F'(x) = f'(2x+2)(2)g(1-x^2) + f(2x+2) \cdot g'(1-x^2)(-2x)$
 $\Rightarrow F'(0) = f'(2)(2)g(1) + f(2)g'(1) = 0 = (4)(2) + 0 = 8$

Answer: B

48. Let $G(x) = f\left(\frac{3x^2+1}{x-1}\right)$. If $f'(0) = 0$, $f'(-1) = 2$ then $G'(0) \dots$

UEE 1997.

- A. -2 B. -1 C. 0 D. 3

Solution: Using chain and quotient rule.

$$G'(x) = f'\left(\frac{3x^2+1}{x-1}\right) \frac{d}{dx}\left(\frac{3x^2+1}{x-1}\right)$$

$$= f'\left(\frac{3x^2+1}{x-1}\right) \frac{(x-1) \frac{d}{dx}(3x^2+1) - (3x^2+1) \frac{d}{dx}(x-1)}{(x-1)^2}$$

$$\Rightarrow G'(x) = f'\left(\frac{3x^2+1}{x-1}\right) \frac{(x-1)6x - (3x^2+1)}{(x-1)^2}$$

$$= f'\left(\frac{3x^2+1}{x-1}\right) \left(\frac{3x^2-6x-1}{(x-1)^2}\right) = f'\left(\frac{3x^2+1}{x-1}\right) \left(\frac{3x^2-6x-1}{(x-1)^2}\right)$$

$$\Rightarrow G'(0) = f'(-1)(-1) = 2(-1) = -2$$

Answer: A

49. Let $f(x) = e^{3x} - \cos^2 x$, then $f'(x) \dots$ UEE

Solution: $f'(x) = 3e^{3x} - 2 \cos x \sin x \leftarrow (e^{3x})' = 3e^{3x}$

50. $\Rightarrow f'(x) = 3e^{3x} - \sin 2x \leftarrow 2 \cos x \sin x = \sin 2x$

The derivative of $f(x) = \sin(x - \cos 2x) \dots$ UEE 1996

Solution: $f'(x) = \frac{d}{dx} \sin(x - \cos 2x) = \frac{d}{dx}(x - \cos 2x) \frac{d}{dx}(2x)$

$$\Rightarrow f'(x) = \cos(x - \cos 2x)(1 + \sin 2x)(2)$$

$$\Rightarrow f'(x) = 2 \cos(x - \cos 2x) + 2 \cos(x - \cos 2x) \sin 2x$$

51. The equation of the line tangent to the graph of

$f(x) = 2x - \sin(1-x)$ at $(1, f(1)) \dots$ UEE

- A. $y = 3x - 5$ B. $y = 3x - 1$ C. $y = x + 1$ D. $y = x - 3$

Solution:

$$f'(x) = \frac{d}{dx}(2x - \sin(1-x)) = 2 - \frac{d}{dx}\sin(1-x) \frac{d}{dx}(1-x)$$

$$\Rightarrow f'(x) = 2 - \cos(1-x)(-1) \Rightarrow f'(1) = 2 + 1 = 3 \text{ and } f(1) = 2$$

$$y - f(1) = f'(1)(x - 1) \Rightarrow y - 2 = 3(x - 1) \Rightarrow y = 3x - 1$$

∴ Answer: B

52. At what does the equation of a line tangent to the graph of $f(x) = e^{2x}$ at $x = 0$ intersects x-axis . . . UEE

A. $\left(-\frac{1}{2}, 0\right)$ B. $(-2, 0)$ C. $\left(\frac{1}{2}, 0\right)$ D. $(\sqrt{e}, 0)$

Solution: $f(x) = e^{2x} \Rightarrow f'(x) = e^{2x}(2) \Rightarrow f'(0) = 2 \Rightarrow y - f(0) = f'(0)(x - 0) \Rightarrow y - 1 = 2x \Rightarrow y = 2x + 1$. The tangent line $y = 2x + 1$ crosses the x-axis when $y = 0 \Rightarrow 2x + 1 = 0$

$$\Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}$$

∴ Answer: A

53. Let $f(x) = x^2\sqrt{2x+12}$, the equation of tangent line to the graph of $f(x)$ at $x = 2$. . . UEE

A. $y - 4 = 16(x - 2)$ C. $y = \frac{1}{\sqrt{12}}(x - 2) - 16$

B. $y = 17(x - 2) + 16$

Solution:

$$\Rightarrow f'(x) = x^2 \frac{d}{dx}\sqrt{2x+12} + \sqrt{2x+12} \frac{d}{dx}x^2 \leftarrow \text{product rule}$$

$$\Rightarrow f'(x) = x^2 \frac{d}{dx}(2x+12)^{\frac{1}{2}} \frac{d}{dx}(2x+12) + \sqrt{2x+12}(2x)$$

$$= \frac{2x^2}{2\sqrt{2x+12}} + 2x\sqrt{2x+12}$$

$$\Rightarrow f'(2) = \frac{8}{8} + 2(2)\sqrt{16} = 1 + 16 = 17 \text{ and } f(2) = 16$$

$$\Rightarrow y - f(2) = f'(2)(x - 2) \Rightarrow y - 16 = 17(x - 2)$$

$$\Rightarrow y = 17(x - 2) + 16$$

Answer: B

If $u = f(x^3)$, $v = g(x^2)$, $f'(x) = \cos x$ and $g'(x) = \sin x$ then find

$$\frac{du}{dv}$$

Solution: $\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv}$ since

$$u = f(x^3) \Rightarrow \frac{du}{dx} = f'(x^3) 3x^2 \text{ and } f'(x^3) = \cos x^3.$$

$$\Rightarrow \frac{du}{dx} = f'(x^3) 3x^2 \Rightarrow \cos(x^3)(3x^2)$$

$$v = g(x^2) \Rightarrow \frac{dv}{dx} = g'(x^2) 2x \text{ and } g'(x^2) = \sin x^2$$

$$\Rightarrow \frac{dv}{dx} = g'(x^2)(2x) \Rightarrow \sin(x^2)(2x) \Rightarrow \frac{dx}{dv} = \frac{1}{(\sin x^2) 2x}$$

$$\therefore \frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv} = \frac{3x^2 \cos x^3}{2x \sin x^2} = \frac{3}{2} x \frac{\cos x^3}{\sin x^2}$$

Let $g(x) = x + 3^x$ and $F(x) = (f \circ g)(x)$ such that $f'(4) = 1$. Then $F'(1)$ is equal to:

A. $1 + 3 \ln 3$

C. $4 + 12 \ln 3$

B. $-1 - 3 \ln 3$

D. $-4 - 12 \ln 3$

Solution: $F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x)) g'(x)$

$$\Rightarrow F'(1) = f'(g(1)) g'(1) = f'(4) (1 + 3 \ln 3) \therefore \text{Answer: B}$$

$$F'(1) = -1 - 3 \ln 3, \text{ since } g'(x) = 1 + 3^x \ln 3 \Rightarrow g'(1) = 1 + 3 \ln 3$$

Let $f(x) = g(\sqrt{4x+1})$. If $g'(3) = \frac{1}{2}$, then $f'(2)$ is

A. $\frac{2}{3}$

B. $\frac{1}{3}$

C. $\frac{\sqrt{5}}{2}$

D. $\frac{\sqrt{13}}{2}$

Solution: $f'(x) = g'(\sqrt{4x+1}) \cdot \frac{d}{dx} \sqrt{4x+1} = \frac{d}{dx} (4x+1) \therefore \text{Answer: B}$

$$\Rightarrow f'(x) = \frac{g'(\sqrt{4x+1})}{2\sqrt{4x+1}} \cdot (4) \Rightarrow f'(2) = \frac{g'(3)(4)}{2\sqrt{9}} = \frac{1}{6} \cdot \frac{4}{3} = \frac{1}{3}$$

57. If $h(x) = \sqrt[3]{g(x)}$, $g(2) = 8$ and $g'(2) = 6$, then the equation of tangent line at $(2, h(2))$ is _____

Solution: $h'(x) = \frac{1}{3}(g(x))^{-\frac{2}{3}} \cdot g'(x)$

$$\Rightarrow h'(2) = \frac{1}{3}(g(2))^{-\frac{2}{3}} g'(2) = \frac{1}{3}[8]^{-\frac{2}{3}} \cdot (6) = \frac{6}{12} = \frac{1}{2}$$

$$\Rightarrow y - h(2) = h'(2)(x - 2) \Rightarrow y - \sqrt[3]{g(2)} = \frac{1}{2}(x - 2) \leftarrow \text{tangent line}$$

$$\Rightarrow y - \sqrt[3]{8} = \frac{1}{2}(x - 2) \Rightarrow y - 2 = \frac{1}{2}x - 1 \Rightarrow y = \frac{1}{2}x + 1$$

58. Let $f(x) = \sin\left(\frac{\pi}{x}\right)$, then gradient (slope) of the curve at $x = 1$ is _____

Solution:

$$\Rightarrow f'(x) = \cos\left(\frac{\pi}{x}\right) \left(-\frac{\pi}{x^2}\right) = f'(1) = \cos \pi \left(-\frac{\pi}{1^2}\right) = \pi \leftarrow \text{slope}$$

59. Let $f(x) = e^{3x-1}$ and $g(x) = \tan x$. If $F(x) = x f(x) + f(g(x))$ then $F'(0)$.

A. $3e^{-1}$ B. $4e^{-1}$ C. $\frac{1}{e} + \frac{1}{e^2}$ D. $\frac{1}{e} + \frac{3}{e^2}$

Solution: $F'(x) = x f'(x) + f(x) + f'(g(x)) g'(x) \Rightarrow F'(0) = (0)$

$f'(0) + f(0) + f'(g(0)) g'(0)$ we have, $f'(x) = 3e^{3x-1}$

$$\Rightarrow f'(0) = 3e^{-1}$$

$$g'(x) = \sec^2 x \Rightarrow g'(0) = \sec^2(0) = 1$$

$$\therefore F'(0) = 0 + e^{-1} + 3e^{-1} = 4e^{-1}$$

60. If $g(x) = x f(x^2 + 2x)$, $f(3) = -2$ and $f'(3) = 5$, then $g'(1)$ is equal to:

A. 20 B. 15 C. 18 D. 12

Solution: $g'(x) = x' f(x^2 + 2x) + x f'(x^2 + 2x) \cdot 2x = 2$

$$\Rightarrow g'(1) = 1 f(3) + 1 f'(3) (4) = -2 + 5(4) = 18$$

61. If $f(3) = 2$, $f'(3) = 4$, $g'(2) = 5$, $g(2) = -1$ and

$F(x) = f(x^2 - 4x) g(3x - 1)$ then $F'(1)$.

A. 22 B. 38 C. 7 D. 19

Answer: B

Answer: C

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Solution: $F'(x) = f'(x^2 - 4x)(2x - 4)g(3x - 1) + f(x^2 - 4x)g'(3x - 1)$
 $\Rightarrow F'(1) = f'(-3)(-2)g(2) + f(-3)g'(2)(3)$

$$\Rightarrow F'(1) = 4(-2)(5) + 2(5)(3) = -10$$

Answer: D

If $f(x) = \ln(1 + \csc x)$, then $f'(x)$ is equal _____

Solution: $f'(x) = \frac{-\cot x(\csc x)}{1 + \csc x} = \frac{-\cot x}{\sin x + 1}$

The equation of the line tangent to the graph of $f(x) = 2x \sin(1 - x)$ at $(1, f(1))$... UEE

A. $y = 3x - 5$ B. $y = 3x - 1$ C. $y = x + 1$ D. $y = x - 3$

Solution: $f'(x) = 2 - \cos(1 - x)(-1) \leftarrow$ using chain rule

$$\Rightarrow f(1) = 2(1) - \sin(1 - 1) = 2 - 0 = 2 \text{ and } f'(1) = 2 - \cos(1 - 1)(-1) = 2 + \cos(0) = 2 + 1 = 3$$

$$\therefore y - f(1) = f'(1)(x - 1) \Rightarrow y - 2 = 3(x - 1) \leftarrow \text{Equation of tangent line.}$$

$$\therefore y = 3x - 1 \text{ Therefore,}$$

Answer: B

Let $f(x) = \begin{cases} 3x - 1, & x \leq 2 \\ x + 3, & x > 2 \end{cases}$ then which of the following true. . .

UEE

A. $f'(2) = 1$

C. $f'(c) = 3$

B. $f'(2) = 5$

D. f has no derivative at $x = 2$

Solution: $f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$ provided the limit exist.

$$f'_+(2) = \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{x + 3 - 5}{x - 2} = \lim_{x \rightarrow 2^+} \frac{x - 2}{x - 2} = 1 \text{ and}$$

$$f'_-(2) = \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{3x - 1 - 5}{x - 2} = \lim_{x \rightarrow 2^-} \frac{3(x - 2)}{x - 2} = 3$$

Thus, $f'_+(2) \neq f'_-(2) \Rightarrow f'(2)$ does not exist.

Answer: D

Let $f(x) = \begin{cases} 3x - 1, & x \leq 2 \\ x^2 + 1, & x > 2 \end{cases}$ then the derivative of $f(x)$ at $x = 2$...

UEE

A. 3

B. 4

C. 5 D. Doesn't exist

Solution: Exercise left for you.

Differentiation using method of logarithm

Differentiation using method of logarithm makes easier, that a function is composed of product, quotients and power. To apply, take the natural logarithm of both sided, then use its properties of logarithm of both sided, then use its properties of logarithm. For example:

$$\text{i) } y = \frac{1}{x} \Rightarrow \ln y = \ln \left(\frac{1}{x} \right) = \ln 1 - \ln x$$

$$\text{ii) } y = x^z \Rightarrow \ln y = \ln x^z = z \ln x$$

Illustrative Example

66. Find $\frac{dy}{dx}$:

$$\text{a. } y = \sqrt{1+x^5}$$

$$\text{e. } y = (ax+b)^n, n \in \mathbb{R}$$

$$\text{b. } y = \sqrt{\frac{x^2}{x^3+4}}$$

$$\text{f. } y = x^3 \sqrt{1-x^2}$$

$$\text{c. } y = \left(\frac{2x+4}{5-6x} \right)^9$$

$$\text{g. } y = e^x (1-x^2) \sin x$$

$$\text{d. } y = \frac{2x^3}{\sqrt{1+x^6}}$$

Solution: Take the natural logarithm ($\ln$) of both side of expression, then apply the property and then differentiate. Finally solve for $\frac{dy}{dx}$.

$$\text{a. } y = \sqrt{1+x^5} \Rightarrow \ln y = \ln \sqrt{1+x^5} \Leftrightarrow \ln y = \frac{1}{2} \ln(1+x^5)$$

$$\Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx} \frac{1}{2} \ln(1+x^5) \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{5x^4}{2(1+x^5)} \quad \text{solving}$$

$$\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y \left(\frac{5x^4}{2(1+x^5)} \right) = \frac{\sqrt{1+x^5} (5x^4)}{2(1+x^5)} = \frac{5x^4}{2\sqrt{1+x^5}}$$

$$d. y = \sqrt{\frac{x^2}{x^3+4}} \Rightarrow \ln y = \frac{1}{2} \ln \left(\frac{x^2}{x^3+4} \right) = \frac{1}{2} (\ln x^2 - \ln(x^3+4))$$

$$\Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx} \left(\frac{1}{2} (\ln x^2 - \ln(x^3+4)) \right)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} - \frac{3x^2}{x^3+4} \Rightarrow \frac{dy}{dx} = y \left(\frac{1}{x} - \frac{3x^2}{x^3+4} \right) \leftarrow \text{solving for}$$

$$\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \sqrt{\frac{x^2}{x^3+4}} \left[\frac{1}{x} - \frac{3x^2}{x^3+4} \right]$$

$$e. y = \left(\frac{2x+4}{5-6x} \right)^9 \Rightarrow \ln y = 9 \ln \left(\frac{2x+4}{5-6x} \right) = 9 [\ln(2x+4) - \ln(5-6x)]$$

$$\Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx} [\ln(2x+4) - \ln(5-6x)]$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 9 \left[\frac{2}{2x+4} + \frac{6x}{5-6x} \right]$$

$$\Rightarrow \frac{dy}{dx} = y \left[\frac{18}{2x+4} + \frac{54}{5-6x} \right] \leftarrow \text{solving for } \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{2x+4}{5-6x} \right)^9 \left[\frac{18}{2x+4} + \frac{54}{5-6x} \right]$$

Exercise left for you.

$$y = (ax+b)^n \Rightarrow \ln y = n \ln(ax+b)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{na}{ax+b} \Rightarrow \frac{dy}{dx} = \frac{y(a)n}{(ax+b)} = (a)(n)(ax+b)^{n-1}$$

$$y = x^3 \sqrt{1-x^3} \Rightarrow \ln y = \ln(x^3 \sqrt{1-x^3})$$

$$\Rightarrow \ln y = \ln x^3 + \frac{1}{2} \ln(1-x^3)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{3}{x} + \frac{-x^2}{1-x^3} \Rightarrow \frac{dy}{dx} = y \left[\frac{3}{x} - \frac{x^2}{1-x^3} \right]$$

$$\Rightarrow \frac{dy}{dx} = x^3 \sqrt{1-x^2} \left[\frac{3}{x} - \frac{x^2}{1-x^3} \right]$$

g. $y = e^x (1+x^2) \sin x \Rightarrow \ln y = \ln (e^x (1+x^2) \sin x)$

$$\Rightarrow \ln y = \ln e^x + \ln (1+x^2) + \ln \sin x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 1 + \frac{2x}{x^2+1} + \frac{\cos x}{\sin x}. \text{ Multiply both side by } y$$

$$\Rightarrow \frac{dy}{dx} = y \left[1 + \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} \right]$$

$$= e^x (1+x^2) \sin x \left[1 + \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} \right]$$

$$\Rightarrow \frac{dy}{dx} = e^x (1+x^2) \sin x + e^x \sin x 2x + (\cos x) e^x (1+x^2)$$

67. Find the equation of the line tangent to the graph of $f(x) = xe^{-\sqrt{x^2+4}}$ at $(0, 0)$.

Solution: Let $y = f(x) \Rightarrow \ln y = \ln x e^{-\sqrt{x^2+4}}$

$$\Rightarrow \ln y = \ln x + \ln e^{-\sqrt{x^2+4}} = \ln x - \sqrt{x^2+4}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{x} - \frac{2x}{2\sqrt{x^2+4}} \Rightarrow \frac{dy}{dx} = y \left(\frac{1}{x} - \frac{x}{\sqrt{x^2+4}} \right)$$

$$\Rightarrow \frac{dy}{dx} = xe^{-\sqrt{x^2+4}} \left(\frac{1}{x} - \frac{x}{\sqrt{x^2+4}} \right) = e^{-\sqrt{x^2+4}} - \frac{x^2 e^{-\sqrt{x^2+4}}}{\sqrt{x^2+4}}$$

So that $\left. \frac{dy}{dx} \right|_{x=0} = e^{-\sqrt{0+4}} - 0 = e^{-2} \leftarrow f'(0)$

$$\Rightarrow y - f(0) = f'(0)(x - 0) \Rightarrow y - 0 = e^{-2}x = y = \frac{x}{e^2} \leftarrow \text{equation of tangent line.}$$

68. Find $\frac{dy}{dx}$ if

a. $y = a^x$

b. $y = x^x$

c. $y = x^{\ln x}$ d. $x^y = e^{x-y}$

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$$\text{Let } y = a^x \Rightarrow \ln y = \ln a^x = x \ln a$$

$$\frac{d}{dx}(x \ln a) \Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln a \Rightarrow \frac{dy}{dx} = y(\ln a) = a^x \ln a$$

$$y = x^x \Rightarrow \ln y = \ln x^x \Rightarrow \ln y = x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln x \frac{d}{dx} x + x \frac{d}{dx} \ln x = \ln x + 1 \quad \text{solving for}$$

$$\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y(\ln x + 1) = x^x (\ln x + 1)$$

$$y = x^{\ln x} \Rightarrow \ln y = \ln(x^{\ln x}) = \ln x \ln x$$

$$\Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx}(\ln x \ln x)$$

• Multiply both side by y .

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{\ln x}{x} + \frac{\ln x}{x} \right) = x^{\ln x} \left(\frac{2 \ln x}{x} \right) = 2 \ln x (x^{\ln x - 1})$$

$x = e^{x-1}$ ← Exercise left for you.

Higher order derivative of function (The n^{th} derivative of function)

If f is differentiable at each number x is then

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \text{ exist} \leftarrow 1^{\text{st}} \text{ derivative.}$$

If f' is differentiable at each x then.

$$f''(x) = \lim_{t \rightarrow x} \frac{f'(t) - f'(x)}{t - x} \text{ exist} \leftarrow f \text{ is twice differentiable (2}^{\text{nd}} \text{ derivative)}$$

Let $f^{(n)}$ be the n^{th} derivative of f at x

$$f^{(n)}(x) = \lim_{t \rightarrow x} \frac{f^{(n-1)}(t) - f^{(n-1)}(x)}{t - x} \text{ exist} \leftarrow n \text{ times differentiable.}$$

Notation

then we write:

$$f'(x) = \frac{dy}{dx} \leftarrow \text{first derivative}$$

- $f''(x) = \frac{d^2 y}{dx^2} \leftarrow$ second-derivative
- $f'''(x) = \frac{d^3 y}{dx^3} \leftarrow$ third derivative
- $f^n(x) = \frac{d^n y}{dx^n} \leftarrow$ nth derivative.

Illustrative Example

69. Find $\frac{d^2 y}{dx^2}$

a. $y = \ln \left(\frac{x}{x^2 + 4x} \right)$

c. $y = \ln (x^2 + x)$

b. $y = \ln \left(\frac{x}{\sqrt{x+4}} \right)$

d. $y = \cos^2 (4x + \pi)$

Solution: Here

a. $y = \ln \left(\frac{x}{x^2 + 4x} \right) \Rightarrow y = \ln x - \ln(x^2 + 4x)$, so that

$$\frac{dy}{dx} = \frac{d}{dx} [\ln x - \ln(x^2 + 4x)] = \frac{1}{x} - \frac{2x+4}{x^2+4x}$$

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} - \frac{2x+4}{x^2+4x} \right) = \frac{d}{dx} \left(\frac{1}{x} \right) - (x^2+4x) \frac{d}{dx} (2x+4)$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{-1}{x^2} - \frac{\left[(x^2+4x) \frac{d}{dx} (2x+4) - \frac{d}{dx} (x^2+4x) \right]}{(x^2+4x)^2}$$

$$\therefore \frac{d^2 y}{dx^2} = -\frac{1}{x^2} - \left[\frac{2x^2 + 8x - (2x+4)(2x+4)}{(x^2+4x)^2} \right]$$

$$\therefore \frac{d^2 y}{dx^2} = -\frac{1}{x^2} + \frac{2x^2 + 8x + 16}{(x^2+4x)^2}$$

b. $y = \ln x - \frac{1}{2} \ln(x+4) \leftarrow \text{properties of logarithm}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} - \frac{1}{2(x+4)} \leftarrow \text{first derivative}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{x^2} + \frac{2}{(2x+8)^2}$$

c. $y = \ln(x^2 + x) \Rightarrow \frac{dy}{dx} = \frac{2x+1}{(x^2+x)}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{2x+1}{(x^2+x)} \right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x^2 + 2x - (2x+1)^2}{(x^2+x)^2} = \frac{-2x^2 - 2x - 1}{(x^2+x)^2}$$

d. $y = \cos^2(4x+1) \Leftrightarrow y = (\cos(4x+1))^2$

$$\frac{dy}{dx} = \frac{d}{dx} (\cos(4x+1))^2 = 2 \cos(4x+1) \frac{d}{dx} \cos(4x+1) \frac{d}{dx} (4x+1)$$

$$\frac{dy}{dx} = -4 (2 \cos \sin(4x+1)) = -4 \cos(4x+1) = -4 \sin(8x+2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} (-4 \sin(8x+2)) = -4 \cos(8x+2) \frac{d}{dx} (8x+2)$$

$$= -32 \cos(8x+2)$$

70. Find: i) $f'(0)$ ii) $f''(0)$

a. $f(x) = x|x|$

b. $f(x) = x^{\frac{5}{3}}$

Solution: a) $f(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases} \Rightarrow f'(x) = 2|x|, x \in (-\infty, \infty)$$

$$\Rightarrow f''(x) = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases} \text{ , Hence}$$

$$\text{i) } f'(0) = 0$$

$$\text{ii) } f''(0) \text{ does not exist.}$$

$$\text{Since } \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x - 0} \text{ doesn't exist}$$

$$\text{b. } f(x) = x^{\frac{5}{3}} \Rightarrow f'(x) = \frac{5}{3} x^{\frac{2}{3}} \Rightarrow \text{i) } f'(0) = 0$$

$$\Rightarrow f''(x) = \left(\frac{5}{3}\right)\left(\frac{2}{3}\right)x^{-\frac{1}{3}} \Rightarrow \text{ii) } f''(0) \text{ doesn't exist.}$$

Note: f'' is not differentiable even if f' is differentiable at $x = 0$.

71. Let $f(x) = x^2|x|$, then find: a) $f'(0)$ b) $f''(0)$ c) $f'''(0)$

Solution:

$$f(x) = x^2|x| = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 3x^2, & x \geq 0 \\ -3x^2, & x < 0 \end{cases} \Rightarrow f''(x) = \begin{cases} 6x, & x \geq 0 \\ -6x, & x < 0 \end{cases}$$

$$\Rightarrow f''(x) = 6|x| \Rightarrow f'''(x) = \begin{cases} 6, & x > 0 \\ \text{when } x = 0 \\ -6, & x < 0 \end{cases}$$

$$\text{a. } f'(0) = 0$$

$$\text{b. } f''(0) = 0$$

$$\text{c. } f'''(0) \text{ doesn't exist because } \lim_{x \rightarrow 0} \frac{f''(x) - f''(0)}{x} \text{ does not exist}$$

$$\text{(i.e., } f'_+(0) = 6 \text{ and } f'_-(0) = -6)$$

72. Find the n th derivative of each function.

$$\text{a. } f(x) = e^{3x+4}$$

$$\text{c. } f(x) = \frac{1}{1-y}$$

$$\text{b. } f(x) = xe^{x^2}$$

$$\text{d. } f(x) = \sin x$$

Solution: Here,

$$f(x) = e^{3x+4} \Rightarrow f'(x) = 3e^{3x+4} \Rightarrow f''(x) = 3^2 e^{3x+4}$$

$$f(x) = e^{3x} \Rightarrow f'(x) = 3e^{3x} \Rightarrow f''(x) = 3^2 e^{3x}$$

$$f(x) = xe^{-x} \Rightarrow f'(x) = x'e^{-x} + x(e^{-x})' = e^{-x} - xe^{-x} = e^{-x}(1-x)$$

$$f'(x) = e^{-x}(1-x) \Rightarrow f''(x) = (1-x)'e^{-x} + (1-x)(e^{-x})' = -e^{-x} + (1-x)(-e^{-x}) = e^{-x}(2-x)$$

$$f''(x) = e^{-x}(2-x)$$

$$\therefore f''(x) = e^{-x}(n-x), \text{ where } n \text{ is positive integer.}$$

$$f(x) = \frac{1}{1-x} \Rightarrow f'(x) = \frac{1}{(1-x)^2} \Rightarrow f''(x) = \frac{2(1-x)}{(1-x)^4} = \frac{2}{(1-x)^3}$$

$$\Rightarrow f''(x) = \frac{(2)(3)}{(1-x)^3} = \frac{6}{(1-x)^3}$$

$$f''(x) = \frac{n!}{(1-x)^{n+1}}, n \text{ is positive integer.}$$

Let $f(x) = \cos x$, then find $f''(x)$.

Solution: Exercise left for you.

Conceptual Example

72b. Let $f(x)$ be a real valued continuous function such that $f(x) \geq 0$

for all x , $f(3) = 3$ and $f'(3) = 1$. Find $\lim_{x \rightarrow 3} \frac{\sqrt{f(x)} - \sqrt{3}}{\sqrt{x} - \sqrt{3}}$.

Explanation: Rationalize both numerator and denominator.

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow 3} \frac{\sqrt{f(x)} - \sqrt{3}}{\sqrt{x} - \sqrt{3}} &= \lim_{x \rightarrow 3} \frac{\sqrt{f(x)} - \sqrt{3}}{\sqrt{x} - \sqrt{3}} \cdot \frac{\sqrt{f(x)} + \sqrt{3}}{\sqrt{f(x)} + \sqrt{3}} \cdot \frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}} \\ &= \lim_{x \rightarrow 3} \left[\frac{f(x) - 3}{x - 3} \cdot \frac{\sqrt{x} + \sqrt{3}}{\sqrt{f(x)} + \sqrt{3}} \right] \\ &= f'(3) \cdot \frac{\sqrt{3} + \sqrt{3}}{\sqrt{3} + \sqrt{3}} = (1)(1) = 1 \end{aligned}$$

73. If $g(x) = f(4x+6)$, then find $g''(x)$.

Solution: $g'(x) = f'(4x+6)(4)$

$$\Rightarrow g''(x) = f''(4x+6)(4)^2 \Rightarrow g'''(x) = f'''(4x+6)(4)^3$$

74. ... $g''(x) = f''(4x+6) \cdot 4^n$

If $G(x) = f(x^2 - 3x)$, $f'(-2) = 3$ and $f''(-2) = 1$ then $G''(1)$ is equal to

- A. -2 B. 10 C. 5 D. 7

Solution: $\Rightarrow G'(x) = f'(x^2 - 3x)(2x - 3) - (2x - 3)f''(x^2 - 3x) \leftarrow$ chain rule

$\rightarrow G''(x) = (2x - 3)f'''(x^2 - 3x)(2x - 3) + f'(x^2 - 3x)(2) \leftarrow$ product rule

$\Rightarrow G'(1) = (-1)f''(-2)(-1) + f'(-1)(2) \leftarrow$ evaluating at $x = 1$

$$\therefore G''(1) = 3 + (1)(2) = 5$$

Answer: C

75. If $f(x) = 5^{ax}$ then $f'''(0) =$ _____ UEE

- A. π^3 B. $(\pi \ln 5)^3$ C. $(\ln 5)^3$ D. $\pi(\ln 5)^3$

Solution: Remember that: $(a^x)' = a^x \ln a$

$$\Rightarrow f'(x) = 5^{ax} (\pi \ln 5) \Rightarrow f''(x) = 5^{ax} (\pi \ln 5)^2$$

$$\Rightarrow f'''(x) = 5^{ax} (\pi \ln 5)^3 \Rightarrow f'''(0) = 5^{a \cdot 0} (\pi \ln 5)^3 = (\pi \ln 5)^3$$

Remark $f'''(0) = (\pi \ln 5)^3 \leftarrow$ nth derivative at $x = 0$

76. Let $f(x) = e^{-2x}$, then $f^{(n)}(0)$ equal to:

- A. 2^n B. $(-2)^n$ C. -2^n D. $(-1)^{n+1} 2^n$

Solution: $f'(x) = e^{-2x} (-2) \Rightarrow f''(x) = e^{-2x} (-2)(-2)$

$$\therefore f^{(n)}(x) = (-2)^n e^{-2x} \Rightarrow f^{(n)}(0) = (-2)^n$$

Answer: B

77. Let $f(x) = \ln(x^2 - 3x)$, then $f''(-1)$ equal to:

- A. $-\frac{5}{16}$ B. $-\frac{17}{16}$ C. $-\frac{5}{16}$ D. $-\frac{19}{16}$

$$\text{Solution: } f'(x) = \frac{2x-3}{x^2-3x} \Rightarrow \frac{d}{dx} f'(x) = \frac{d}{dx} \left(\frac{2x-3}{x^2-3x} \right)$$

$$\Rightarrow f''(x) = \frac{(x^2-3x) \frac{d}{dx} (2x-3) - (2x-3) \frac{d}{dx} (x^2-3x)}{(x^2-3x)^2}$$

$$f''(x) = \frac{2(x^2-3x) - (2x-3)^2}{(x^2-3x)^2} = \frac{-2x^2 + 6x - 9}{(x^2-3x)^2}$$

$$\Rightarrow f''(-1) = \frac{-2 + -6 - 9}{(1+3)^2} = \frac{-17}{16}$$

Answer: B

78. Let $f(x) = \log x^2$, then $f'''(x)$ equal to: _____

$$\text{Solution: } f(x) = \log x^2 = 2 \log x = \frac{2 \log_e x}{\log_e 10} = \frac{2 \ln x}{\ln 10}$$

679

$$\Rightarrow f'(x) = \frac{2}{x \ln 10} \Rightarrow f''(x) = \frac{-2}{x^2 \ln 10}$$

$$\Rightarrow f''(x) = \frac{4}{x^3 \ln 10} = \frac{4}{x^3} \log_{10}^e \leftarrow 3^{\text{rd}} \text{ derivative}$$

79. If $G(x) = f(x^2 + x)$, $f'(2) = -2$ and $f''(2) = 3$ then $G''(1)$ equal to:
UEE

A. -18 B. 9 C. 23 D. 27

Solution: $G'(x) = f'(x^2 + x)(2x + 1) \leftarrow$ chain rule

$$\Rightarrow G''(x) = (2x + 1)f''(x^2 + x)(2x + 1) + f'(x^2 + x)(2)$$

$$\Rightarrow G''(x) = (2x + 1)^2 f''(x^2 + x) + f'(x^2 + x)(2) \text{ product rule}$$

$$\Rightarrow G''(1) = (9)f''(2) + f'(2)(2) \leftarrow \text{equating at } x = 1$$

$$\Rightarrow G''(1) = 9(3) + (-2)(2) = 23$$

Answer: C

80. If $f(x) = xe^{3x}$, then $f''(x)$ equal to _____

Solution: Exercise left for you.

81. If $f(x) = \frac{1}{x+3}$, then find $f''(x) =$ _____

Solution: $\leftarrow$ exercise left for you. $\left(\text{Ans. } \frac{(n!)(-1)^n}{(x+3)^{n+1}} \right)$

Implicit Differentiation

$y = f(x) \leftarrow$ explicit function

$f(x, y) = x^2 + y^2 \leftarrow$ implicit function

The steps to differentiate, implicit function:

- Differentiate both side with respect to x .
- Collect all terms involving $\frac{dy}{dx}$ and solve for $\frac{dy}{dx}$.

Illustrative Example

82. Find: i) $\frac{dy}{dx}$

ii) Slope and equation of tangent line. iii) $\frac{dx}{dy}$

a. $x^2 + y^2 = 5$, at $(-1, 2)$

c. $y^2 + y = x^2$, at $(\sqrt{2}, 1)$

b. $x^2 + y^2 = 10$ at $(3, 1)$ d. $x = \sin y$ at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$

Solution:

a. $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}5 = 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

← Rate of change of y with respect to x

ii) Slope = $\left. \frac{dy}{dx} \right|_{x=-1, y=2} = \frac{-x}{y} = \frac{1}{2}$ ← Equation of tangent line to circle

$\Rightarrow y - 2 = \frac{1}{2}(x + 1) \Rightarrow y = \frac{1}{2}x + \frac{5}{2}$ ← Equation of tangent line

iii) $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dy}(5) \Rightarrow 2x \frac{dx}{dy} + 2y = 0$

$\Rightarrow \frac{dx}{dy} = -\frac{2y}{2x} = \frac{-y}{x}$ ← rate of change of x with respect to y

b. Exercises left for you.

c.i) $\frac{d}{dx}(y^2 + y) = \frac{d}{dx}x^2 \Rightarrow 2y \frac{dy}{dx} + \frac{dy}{dx} = 2x$

$\Rightarrow \frac{dy}{dx}(2y + 1) = 2x$ ← factoring $\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{2y + 1}$ ← Rate of change of y with respect to x

ii) Slope, $\left. \frac{dy}{dx} \right|_{x=\sqrt{2}, y=1} = \frac{2\sqrt{2}}{2+1} = \frac{2\sqrt{2}}{3}$ ← slope

$y - 1 = \frac{2\sqrt{2}}{3}(x - \sqrt{2})$ ← Equation of tangent line

iii) $\frac{d}{dy}(y^2 + y) = \frac{d}{dy}x^2 \Rightarrow 2y + 1 = 2x \frac{dx}{dy}$ ← solving for $\frac{dx}{dy}$

$$\Rightarrow \frac{dx}{dy} = \frac{2y+1}{2x} \leftarrow \text{rate of change of } x \text{ with respect to } y$$

$$d. \text{ i) } \frac{d}{dx} x = \frac{d}{dx} \sin y \Rightarrow 1 = \cos y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} \Rightarrow \left. \frac{dy}{dx} \right|_{x=\frac{1}{2}, y=\frac{\pi}{6}} = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{2}{\sqrt{3}}$$

$$\text{ii) } y - \frac{\pi}{6} = \frac{2}{\sqrt{3}} \left(x - \frac{1}{2} \right) \Rightarrow y = \frac{2x}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{\pi}{6} \leftarrow \text{tangent line}$$

$$\text{iii) } \frac{d}{dy} x = \frac{d}{dy} \sin y \Rightarrow \frac{dx}{dy} = \cos y$$

$$B. \text{ Let } F(x) = e^{-x} \ln x. \text{ Find: a) } F''(x) \quad \text{b) } F''(1)$$

Solution: Let $f(x) = e^{-x}$ and $g(x) = \ln x \Rightarrow F(x) = f(x)g(x) \Rightarrow F'(x) = f'(x)g(x) + f(x)g'(x) \leftarrow \text{product rule}$

$$F''(x) = (f'(x)g(x) + f(x)g'(x))' = f''(x)g(x) + 2f'(x)g'(x) + g''(x)f(x)$$

$$\text{a. } F''(x) = e^{-x} \ln x - \frac{2e^{-x}}{x} - \frac{e^{-x}}{x^2}$$

$$\text{b. } F''(1) = e^{-1}(0) - 2e^{-1} - e^{-1} = -3e^{-1}$$

Explanation

$$g(x) = \ln x \Rightarrow g'(x) = \frac{1}{x}, g''(x) = -\frac{1}{x^2}$$

$$f(x) = e^{-x} \Rightarrow f'(x) = -e^{-x}, f''(x) = e^{-x}$$

$$84. \text{ Let } f(x) = x^3 e^{-x}, \text{ Find a) } f''(x) \quad \text{b) } f''(1)$$

$$\text{Solution: } f'(x) = 3x^2 e^{-x} - x^3 e^{-x}$$

$$\text{a) } f''(x) = (3x^2 e^{-x} - x^3 e^{-x})' = 6x e^{-x} - 6x^2 e^{-x} + x^3 e^{-x}$$

$$\text{b) } f''(1) = 6e^{-1} - 6e^{-1} + e^{-1} = e^{-1}$$

$$85. \text{ Let } f(x) = 3^{x-2}, \text{ then find a) } f''(x) \quad \text{b) } f''(2)$$

$$\text{Solution: } f'(x) = 3^{x-2} \ln 3 \Rightarrow f''(x) = 3^{x-2} (\ln 3)^2 \dots f''(x) = 3^{x-2} (\ln 3)^2$$

$$\text{c) } f''(2) = 3^{0} (\ln 3)^2 = (\ln 3)^2$$

Supplementary exercise on Unit – 12

1. Find the equation of the tangent line to the graph of the function at the indicated point.
 - a. $f(x) = \begin{cases} x - \pi, & x \leq 0 \\ 1 - 2 \sin x, & x > 0 \end{cases}$ at $(\pi, f(\pi))$ h. $f(x) = \sin x$ at $x = \frac{\pi}{6}$
 - b. $f(x) = \frac{3x}{x-2}$ at $(1, f(1))$ i. $f(x) = \log_{10} x$; $x = 8$
 - c. $f(x) = x^3 + 2$; $(-1, 0)$ j. $f(x) = 3x - \sin(2x)$ at $x = 2$
 - d. $f(x) = 3x^2 - 2x - 1$; $(2, 7)$ k. $f(x) = \begin{cases} x^3, & x \leq -1 \\ 3x + 2, & x > -1 \end{cases}$
 - e. $f(x) = \frac{x-1}{x+1}$ at $(1, 0)$ l. $f(x) = 3x + \cos(4x)$ at $x = \frac{\pi}{4}$
 - f. $f(x) = \frac{2x+1}{3-x^2}$, at $\left(1, \frac{3}{2}\right)$ m. $f(x) = x - \cos(4x)$ at $x = \frac{\pi}{4}$
 - g. $f(x) = xe^{-\sqrt{x+2}}$ at $(0, 0)$ n. $f(x) = x \cos 3x$ at $x = \frac{\pi}{4}$
2. Let $f(x) = 2x^3 - 6x$, find the set of all x for which.
 - a. tangent line is horizontal d. the slope is negative
 - b. the slope is positive e. the slope is -2
 - c. the slope is 2
3. Let $f(x) = x^3 + 2x^2 - 15x + 2$; find the equation of the tangent line at the point where the graph crosses. a) $y = 2$ b) $y = x^2 - 2$
4. Determine the intervals on which the function is differentiable.
 - a. $f(x) = x^2 - 4x$ f. $f(x) = |x - x^2|$
 - b. $f(x) = \sqrt{x+5}$ g. $f(x) = \begin{cases} 3x & \text{if } x > 1 \\ 4 - x^2 & \text{if } x \leq 1 \end{cases}$
 - c. $f(x) = \sqrt{25 - 9x^2}$ h. $f(x) = \frac{x+2}{5-x}$
 - d. $f(x) = |x - 3|$ i. $f(x) = |x^2 - 3x + 2|$
 - e. $f(x) = |3x - 5|$
5. Find $f'(x)$:
 - a. $f(x) = \log_x 3$ i. $f(x) = \sqrt{x^2 - 5x - 6}$

$$h. f(x) = \log_4 \sqrt{x^3} \quad j. f(x) = e^{\sqrt{x^2+4x}}$$

$$i. f(x) = \log_2 (x^2 + 5) \quad k. f(x) = \frac{x^2 + 3}{\log_2 x}$$

$$d. f(x) = \log_{10} \sqrt{x^2+1} \quad l. f(x) = \sec^2(x-3)$$

$$e. f(x) = \ln \left(\frac{6x+5}{3x^2+x+1} \right) \quad m. f(x) = \frac{1}{5^x + 6}$$

$$f. f(x) = \frac{\cos x}{\sqrt{3x-1}} \quad n. f(x) = 3^{6-x^2}$$

$$g. f(x) = \sin(x^2 + 1) \quad o. f(x) = e^{\sin \sqrt{x+4}}$$

$$h. f(x) = \frac{2x^3}{\sqrt{1+x^4}} \quad p. f(x) = \left(\frac{4x-1}{2-5x} \right)^3$$

Given that: $f'(1) = -2$, $f(1) = 3$, $g'(1) = 5$ and $g(1) = 4$. Then find $F(1)$ if:

$$1. F(x) = f(x)g(x) \quad b. F(x) = \frac{3x+4}{g(x)} \quad c. F(x) = f(x)\sqrt{4x-1}$$

If $f(5) = 3$, $f'(5) = -2$, $g(-2) = 1$, $g'(-2) = 2$ and $F(x) = f(2-3x)$, $g(x^2+3x)$ then find $F'(-1)$.

Let $f(x) = g(\sqrt{5x+1})$, if $g'(4) = \frac{2}{5}$ then find $f'(3)$.

Let $f(x) = e^{4x-1}$, $g(x) = x^2 + 3x$, and $F(x) = x^3 f(x) + f(g(x))$ then find $F'(1)$.

Let $f(x) = \ln(x^2 - 4x)$ then find $f''(-1)$.

If $G(x) = f(x^3 - 5x)$, $f'(4) = -2$ and $f''(4) = 3$ then find $G'(-1)$.

Let $G(x) = f\left(\frac{x}{x-4}\right)$, $f'\left(-\frac{1}{3}\right) = 2$ find $G'(1)$.

Let $f(x) = e^{-4x}$ for each positive integer n then find:

a. $f^n(x)$ b. $f^n(0)$

If $f(x) = xe^x$, then find $f^n(x)$.

15. Let $g(x) = \frac{\ln(x^2 + 4x)}{x^2 + 4x}$, find $g'(1)$.
16. Find the equation of the line tangent to the circle at the indicated point.
 a. $x^2 + y^2 = 5$, $(1, 2)$ b. $x^2 + y^2 - 2x + 6y - 16 = 0$, $(2, -8)$
17. Let $f(x) = x^2(e^x)$ then find $f''(1)$.
18. Let $f(x) = 3^{2x}$ then find $f'''(0)$.
19. If $g(x) = f(2x + 4)$, then $g''(x)$.
20. Find the n th derivative ($f^{(n)}(x)$) for the given function:
 a. $f(x) = \frac{1}{3-x}$ b. $f(x) = \frac{1}{2+x}$



Unit Thirteen

Applications of Differential Calculus

Differential Calculus is important:

- To determine the extreme value of function.
- To determine where the function is increasing or decreasing.
- To determine the curvature (concavity) of the graph.
- To determine the maximization and minimization problem.

In this unit, you will learn how derivative can be used to determine extreme value, monotonicity, concavity of the function.

Extreme values of function:

Definition: Let f be function defined on an interval I containing c .

- i) $f(c)$ is the absolute minimum of f on I if $f(c) \leq f(x)$ for all x in I . (That is $f(c)$ is the smallest value of f on I)
- ii) $f(c)$ is the absolute maximum of f on I if $f(c) \geq f(x)$ for all x in I . (That is $f(c)$ is the largest value of f on I).

The minimum and the maximum value of the function on an interval are called *extreme values* of function.

Note:

- i) Often use the terms instead of saying:
 - Absolute minimum we say minimum.
 - Absolute maximum we say maximum.
- ii) A function may not necessarily have extreme values on an interval.

Example:

- a) $f(x) = 3x + 2$, on $(-2, 4)$ has no extreme value but has extreme value on $[-2, 4]$.
- b) $f(x) = x^3 + 2$ on $(1, 3)$ has no extreme value but has extreme value on $[1, 3]$.

Illustrative Example

1. Let $f(x) = 4x + 1$. Find the maximum and the minimum elements of f in the indicated interval I .

a. $(1, 5)$

b. $[1, 5]$

c. $\{1, 5\}$

Solution: f is continuous on given interval

- a. $f(x) = 4x + 1$ has no minimum and maximum value on $(1, 5)$.
- b. $f(x) = 4x + 1$ has minimum value on $[1, 5)$ which is $f(1) = 5$ but has no maximum value.
- c. $f(x) = 4x + 1$ has maximum value on $(1, 5]$ which is $f(5) = 21$ but has no minimum value on $(1, 5]$.
2. Let $f(x) = \frac{1}{x-2}$, find the maximum and the minimum value of f on given interval:

- a. $[0, 3]$ b. $(2, 5)$ c. $(2, 5]$ d. $[3, 5)$

Solution: Here

- a. $f(x) = \frac{1}{x-2}$ is not continuous on $[0, 3]$ for $x = 2 \Rightarrow f$ has no maximum and minimum value.
- b. $f(x) = \frac{1}{x-2}$ has no maximum and minimum.
- c. $f(x) = \frac{1}{x-2}$ has minimum (smallest value) on $(2, 5]$ which is $f(5) = \frac{1}{3}$ but has no maximum.
- d. Exercise left for you.

Note: Extreme value theorem is an existence theorem because it tells of the existence of minimum and maximum value but does not show how to find these values.

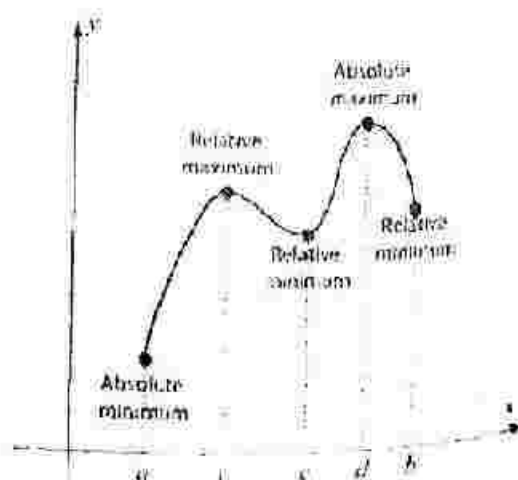
Relative Extreme values and critical number

The relative minimum and relative maximum value is called Relative Extreme value.

Consider the graph on the right

Fig. how maxima and minima are classified
From the graph we observe the following terms.

- i) $f(a)$ is *Absolute minimum*;



No smaller value of f anywhere. Also local minimum.

ii) $f(a)$ is: **Absolute maximum**: No greater value of f anywhere. Also local maximum.

iii) $f(b)$ is: **local maximum**: No larger value of f nearby.

iv) $f(c)$ and $f(b)$ are **local minimum**: No smaller value of f nearby.

Relative maximum is the high point (peak). (That is no greater value of f nearby.)

Relative minimum is the low point (valley). (That is no smaller value of f nearby.)

Definition: A function f is said to have:

i) **Relative maximum** at c if there is an open interval containing c on which $f(c)$ is largest value. That is, $f(c) \geq f(x)$ for all x on the interval.

ii) **Relative minimum** at c if there is an open interval containing c on which $f(c)$ is smallest value. That is, $f(c) \leq f(x)$ for all x on the interval.

Critical Number

Definition: Let c be in the domain of f . If $f'(c) = 0$ or f has no derivative at c , then c is called **critical number** of f .

Finding Extreme Values

The only places where a function f can possibly have an extreme value are:

- at critical point ($f'(c) = 0$ or $f'(c)$ doesn't exist).
- at end points of the interval.

Theorem: Relative extreme only at critical number.

If f has a relative minimum or relative maximum at $x = c$, then c is critical number of f .

Note: The converse of the theorem is not true. That is the critical numbers of a function need not produce relative extrema.

How to find the Absolute Extrema of continuous Function f on a closed interval $[a, b]$.

To find extrema:

- Evaluate f at all critical point of f in (a, b) .
- Evaluate f at each end points of $[a, b]$.
- Take the largest and smallest of these values.

Illustrative Example

3. Find:

i) Critical numbers

ii) Absolute maximum or absolute minimum

a. $f(x) = 2x^3 - 3x^2 - 12x + 7, [-2, 3]$

b. $f(x) = 3x^3 - 18x^2 + 8, [-1, 5]$

c. $f(x) = \frac{8x}{x^2 + 4}, [-4, 4]$

d. $f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}}, [-1, 4]$

e. $f(x) = 2 \sin x + x, [0, 2\pi]$

f. $f(x) = x^6 - 3x^4 + 8, [-2, 2]$

g. $f(x) = x^3(5 - 2x), [-1, 2]$

Solution: Note to find critical number set $f'(x) = 0$.

a. $f(x) = 2x^3 - 3x^2 - 12x + 7 \Rightarrow f'(x) = 6x^2 - 6x - 12$ ex 12

$\Rightarrow f'(x) = 6x^2 - 6x - 12 = 0 \dots$ set $f'(x) = 0$

$\Rightarrow f'(x) = 6(x - 2)(x + 1) = 0 \dots$ factor

$\Rightarrow x = 2, -1 \dots$ critical numbers

The candidates $\{-2, -1, 2, 3\}$. Evaluating f at two critical numbers and at the end point of $[-2, 3]$

• At left end point $f(-2) = 3$.

• At critical numbers $f(-1) = 14$ and $f(2) = -13$.

• At right end point $f(3) = -2$

 $\therefore$ The maximum value of f on $[-2, 3]$ is $f(-1) = 14$ and the minimum value of f on $[-2, 3]$ is $f(2) = -13$.

b. $f(x) = 3x^3 - 18x^2 + 8 \Rightarrow f'(x) = 9x^2 - 36x$ set $f'(x) = 0$
 $\Rightarrow 9x(x - 4) = 0 \Rightarrow x = 0$ or $x = 4$.

Thus, 0 and 4 are the critical numbers. Evaluating f at critical numbers and at end points.

$\Rightarrow f(-1) = -13, f(0) = 8, f(4) = -88$ and $f(5) = -67$

 $\therefore$ The maximum value of f on $[-1, 5]$ is $f(0) = 8$. $\therefore$ The minimum value of f on $[-1, 5]$ is $f(4) = -88$.

c. $f(x) = \frac{8x}{x^2 + 4} \Rightarrow f'(x) = \frac{8(x^2 + 4) - 2x(8x)}{(x^2 + 4)^2} = \frac{32 - 8x^2}{(x^2 + 4)^2}$

$$\Rightarrow f'(x) = \frac{32 - 8x^2}{(x^2 + 4)^2} = 0 \Rightarrow 8x^2 = 32 \Rightarrow x = -2, 2$$

The candidates are, $\{-4, -2, 2, 4\}$. Hence, evaluating f at these points gives

$$\Rightarrow f(-4) = \frac{-8}{5}, f(-2) = -2, f(2) = 2 \text{ and } f(4) = \frac{8}{5}$$

$\therefore f(-2) = -2$ is the absolute minimum value on $[-4, 4]$.

$\therefore f(2) = 2$ is the absolute maximum value on $[-4, 4]$.

$$1. f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}} \Rightarrow f'(x) = \frac{10}{3}x^{-\frac{1}{3}} - \frac{5}{3}x^{\frac{2}{3}} = \frac{5}{3} \left[\frac{2-x}{\sqrt[3]{x}} \right]$$

$$\Rightarrow f'(x) = \frac{5(2-x)}{3\sqrt[3]{x}} = 0 \Rightarrow x = 2$$

Also, $x = 0$ is critical number because $f'(x)$ does not exist. Thus, $x = 0$ and $x = 2$ are critical numbers. The candidates are $\{-1, 0, 2, 4\}$. Hence, evaluating f at these points gives,

$$\Rightarrow f(-1) = 5 - (-1) = 6, f(0) = 0, f(2) \approx 4.76, f(4) \approx 2.52$$

$\therefore f(-1) = 6$ is the absolute maximum on $[-1, 4]$

$\therefore f(0) = 0$ is the absolute minimum of f on $[-1, 4]$.

$$2. f(x) = 2 \sin x + x \Rightarrow f'(x) = 2 \cos x + 1$$

$$\Rightarrow f'(x) = 2 \cos x + 1 = 0 \Rightarrow \cos x = -\frac{1}{2}$$

$$\Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3} \text{ are critical numbers. The candidates are}$$

$$\left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi \right\}. \text{ Hence, evaluating } f \text{ at these points, we get,}$$

$$f(0) = 0, f\left(\frac{2\pi}{3}\right) = \sqrt{3} + \frac{2\pi}{3}, f\left(\frac{4\pi}{3}\right) = -\sqrt{3} + \frac{4\pi}{3}, f(2\pi) = 2\pi$$

$\therefore f(2\pi) = 2\pi \leftarrow$ Absolute maximum

$\therefore f(0) = 0 \leftarrow$ Absolute minimum

$$f. f(x) = x^6 - 3x^4 + 8 \Rightarrow f'(x) = 6x^5 - 12x^3$$

$$\Rightarrow f'(x) = 6x^3(x^2 - 2) = 0 \Rightarrow x = 0, x = \pm\sqrt{2}$$

$\therefore x = -\sqrt{2}, 0, \sqrt{2}$. The candidates are $\{-2, -\sqrt{2}, 0, \sqrt{2}, 2\}$. Hence, evaluating f at these values, we get, $f(-2) = 24$,

$$f(-\sqrt{2}) = 4, f(0) = 8, f(\sqrt{2}) = 4 \text{ and } f(2) = 24$$

$\therefore f(-2) = f(2) = 24$ is Absolute maximum and

$f(-\sqrt{2}) = f(\sqrt{2}) = 4$ is Absolute minimum.

$$g. f(x) = x^{\frac{2}{3}}(5-2x) = 5x^{\frac{2}{3}} - 2x^{\frac{5}{3}} \Rightarrow f'(x) = \frac{10}{3}x^{-\frac{1}{3}} - \frac{10}{3}x^{\frac{2}{3}}$$

$$\Rightarrow f'(x) = \frac{10}{3} \left[\frac{1-x}{3\sqrt[3]{x}} \right] = 0 \Rightarrow x-1=0 \Rightarrow x=1. \text{ Also } x=0 \text{ is a}$$

critical number since $f'(0)$ does not exist.

$\therefore x=0, x=1$ are the critical numbers.

Hence, $f(0) = 0$ is Absolute minimum, $f(-1) = 7$ is Absolute maximum, $f(1) = 3$, $f(2) = \sqrt[3]{4}$

Absolute maximum and minimum on interval $(-\infty, \infty)$

i) If $\lim_{x \rightarrow -\infty} f(x) = +\infty = \lim_{x \rightarrow +\infty} f(x)$ then f has absolute minimum but no absolute maximum on $(-\infty, \infty)$.

ii) If $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x) = -\infty$ then f has absolute maximum on $(-\infty, \infty)$ but no absolute minimum.

iii) If $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow +\infty} f(x) = +\infty$ or $\lim_{x \rightarrow -\infty} f(x) = +\infty$ and $\lim_{x \rightarrow +\infty} f(x) = -\infty$, then f has neither an absolute maximum nor an absolute minimum.

Illustrative Example

4. Find the absolute maximum, the absolute minimum and the range of f on indicated interval.

a. $f(x) = 4x^3 + 3x^4, (-\infty, \infty)$

d. $f(x) = x^3 - 4x + 2, (-\infty, \infty)$

b. $f(x) = x^2 - 3x - 1, (-\infty, \infty)$

e. $f(x) = \frac{1}{x^2 - 3x}$ on $(0, 3)$

c. $f(x) = 3 - 6x - 3x^2, (-\infty, \infty)$

Solution:

a. $\lim_{x \rightarrow \pm\infty} (4x^3 + 3x^4) = +\infty \Rightarrow f$ has absolute minimum

$$\Rightarrow f'(x) = 12x^2 + 12x^3 = 0 \Rightarrow 12x^2(1+x) = 0 \Rightarrow x = 0 \text{ and } x = -1$$

$$\therefore f(0) = 0 \text{ and } f(-1) = -4 + 3 = -1 \text{ is the smallest value.}$$

$$\therefore \text{Range of } f = [-1, \infty)$$

b. $\lim_{x \rightarrow \pm\infty} (x^2 - 3x - 1) = +\infty \leftarrow f$ has absolute minimum

$$\Rightarrow f'(x) = 2x - 3 = 0 \Rightarrow x = \frac{3}{2} \text{ is the critical number}$$

$$\therefore f\left(\frac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} - 1 = -\frac{13}{4} \text{ is the smallest value.}$$

$$\therefore \text{Range of } f = \left[-\frac{13}{4}, \infty\right)$$

c. $\lim_{x \rightarrow \pm\infty} (3 - 6x - 3x^2) = -\infty \leftarrow$ absolute maximum

$$\Rightarrow f'(x) = -6 - 6x = 0 \Rightarrow x = -1 \text{ is critical number}$$

$$\Rightarrow f(-1) = 3 + 6 - 3 = 6 \Rightarrow \text{the largest value on } (-\infty, \infty) \text{ is } 6.$$

$$\therefore \text{Range of } f = (-\infty, 6].$$

d. $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} x^3 - 4x + 2 = -\infty$ but $\lim_{x \rightarrow +\infty} x^3 - 4x + 2 = \infty$.

$$\therefore f \text{ has no absolute extrema.}$$

e. Since $\lim_{x \rightarrow 0^+} \frac{1}{x(x-3)} = -\infty$ and $\lim_{x \rightarrow 3^-} \frac{1}{x(x-3)} = \frac{1}{0^-} = -\infty$.

$$\therefore f \text{ has absolute maximum on } (0, 3).$$

$$\Rightarrow f'(x) = \frac{-2x+3}{(x^2-3x)} = 0 \Rightarrow -2x+3=0 \Rightarrow x = \frac{3}{2}$$

$$\Rightarrow f\left(\frac{3}{2}\right) = \frac{1}{\frac{9}{4} - \frac{9}{2}} = -\frac{4}{9} \text{ is largest value on } (0, 3)$$

$$\therefore \text{Range of } f = \left(-\infty, -\frac{4}{9}\right]$$

Critical number of Absolute value function

Illustrative Example

5. Find the critical number of each of the following:

- a. $f(x) = |x - 2|$ d. $f(x) = |x^2 - 9|$
 b. $f(x) = |x + 3| + 1$ e. $f(x) = |x^2 - 6x + 8|$
 c. $f(x) = 3 - |x - 1|$ f. $f(x) = |x^2 - 1|$

Solution:

$$a) f(x) = |x - 2| = \begin{cases} x - 2 & \text{if } x > 2 \\ 0 & \text{if } x = 2 \\ -x + 2 & \text{if } x < 2 \end{cases} \Rightarrow f'(x) = \begin{cases} 1 & \text{if } x > 2 \\ \text{DNE} & \text{if } x = 2 \\ -1 & \text{if } x < 2 \end{cases}$$

 $\therefore x = 2$ is critical number since $f'(2)$ does not exist.

$$b) f(x) = |x + 3| + 1 = \begin{cases} x + 4 & \text{if } x > -3 \\ 1 & \text{if } x = -3 \\ -x - 2 & \text{if } x < -3 \end{cases} \Rightarrow f'(x) = \begin{cases} 1 & \text{if } x > -3 \\ \text{DNE} & \text{if } x = -3 \\ -1 & \text{if } x < -3 \end{cases}$$

 $\therefore x = -3$ is critical number since $f'(-3)$ does not exist.

$$c) f(x) = 3 - |x - 1| = \begin{cases} 4 - x & \text{if } x > 1 \\ 3 & \text{if } x = 1 \\ x + 2 & \text{if } x < 1 \end{cases} \Rightarrow f'(x) = \begin{cases} -1 & \text{if } x > 1 \\ \text{DNE} & \text{if } x = 1 \\ 1 & \text{if } x < 1 \end{cases}$$

 $\therefore x = 1$ is critical number since $f'(1)$ does not exist.

$$d) f(x) = |x^2 - 9| = \begin{cases} x^2 - 9 & \text{if } x < -3 \\ 0 & \text{if } x = -3 \text{ and } x = 3 \\ 9 - x^2 & \text{if } -3 < x < 3 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x & \text{if } x < -3 \text{ and } x > 3 \\ \text{DNE} & \text{if } x = -3 \text{ and } x = 3 \\ -2x & \text{if } -3 < x < 3 \end{cases}$$

$\Rightarrow f'(x) = 0 \Rightarrow x = 0$ and $x = -3$ and $x = 3$ are critical number because $f'(-3)$ and $f'(3)$ does not exist.

 $\therefore \{-3, 0, 3\}$ are critical numbers.

$$e) f(x) = |x^2 - 6x + 8| = \begin{cases} x^2 - 6x + 8 & \text{if } x < 2 \text{ and } x > 4 \\ 0 & \text{if } x = 2 \text{ and } x = 4 \\ -x^2 + 6x - 8 & \text{if } 2 < x < 4 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 2x-6 & \text{if } x < 2 \text{ and } x > 4 \\ \text{DNE} & \text{if } x = 2 \text{ and } x = 4 \\ -2x+6 & \text{if } 2 < x < 4 \end{cases}$$

Hence $x = 2$ and $x = 4$ are critical number since $f'(2)$ and $f'(4)$ does not exist and $f'(x) = 0 \Rightarrow 2x - 6 = 0 \Rightarrow x = 3$.

$\therefore \{2, 3, 4\}$ are critical number.

f) $f(x) = |x^2 - 1| \leftarrow$ Exercise left for you. Answer: $\{-1, 0, 1\}$

Rolle's theorem and the mean value theorem

Rolle's theorem

Let f be function that satisfies all the following three conditions.

- i) f is continuous on the closed interval $[a, b]$.
- ii) f is differentiable on the open interval (a, b) .
- iii) $f(a) = f(b)$.

Then there exist a number c in (a, b) such that $f'(c) = 0$.

Note: Rolle's theorem tells us the existence of horizontal tangent line

Illustrative Example

6. Find all number c in $[a, b]$ that satisfies the conclusion of Rolle's Theorem.

- a. $f(x) = \frac{x^3}{3} - 3x$, on $[-3, 3]$
- d. $f(x) = 3 - |x - 2|$ on $[0, 4]$
- b. $f(x) = x^4 - 2x^2$, on $[-2, 2]$
- e. $f(x) = x\sqrt{x+4}$, on $[-4, 0]$
- c. $f(x) = \sin \pi x$, on $[-2, 2]$
- f. $f(x) = x^3 - 3x + 2$

Solution: Recall Rolle's theorem

a) $f(x) = \frac{x^3}{3} - 3x$ is continuous on $[-3, 3]$ and differentiable on

$(-3, 3)$. Moreover, since $f(-3) = 0 = f(3)$, there exist c in $(-3, 3)$ such that $f'(c) = 0$.

$$\therefore f'(x) = x^2 - 3 \Rightarrow f'(c) = c^2 - 3 = 0 \Rightarrow c^2 = 3 \Rightarrow c = \pm\sqrt{3}.$$

b) $f(x) = x^4 - 2x^2$ is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. Moreover $f(-2) = 0 = f(2)$, so that there exist c in $(-2, 2)$ such that $f'(c) = 0 \Rightarrow f'(x) = 4x^3 - 4x$

$$\Rightarrow f'(c) = 4c^3 - 4c = 0 \Rightarrow 4c(c^2 - 1) = 0 \Rightarrow c = 0, -1, 1.$$

$\therefore$ The graph has 3 horizontal tangent lines.

c) $f(x) = \sin \pi x$ is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. Moreover, since $f(-2) = f(2) = 0$, there exist c on $(-2, 2)$ such that $f'(c) = 0$. $\Rightarrow f'(x) = (\cos \pi x) \pi = \pi \cos \pi x = 0$

$$\Rightarrow x = -\frac{1}{2} \text{ and } x = \frac{1}{2} \quad \therefore c = -\frac{1}{2} \text{ and } c = \frac{1}{2}$$

d) $f(x) = 3 - |x - 2|$ is continuous on $[0, 4]$ but it is not differentiable on $(0, 4)$.

$\therefore$ Rolle's theorem is not applicable.

e) $f(x) = x\sqrt{x+4}$ is continuous on $[-4, 0]$ and differentiable on $(-4, 0)$. Moreover $f(-4) = f(0) = 0$, so that there exist c in $(-4, 0)$

$$\text{such that } f'(c) = 0. \Rightarrow f'(x) = \frac{3x+8}{2\sqrt{x+4}} = 0 \Rightarrow x = -\frac{8}{3}$$

$$\therefore c = -\frac{8}{3} \Rightarrow \text{the graph has horizontal tangent line.}$$

f) $f(x) = x^2 - 3x + 2$ ← Exercise left for you.

Mean value theorem

The mean value theorem which has first stated by Joseph-Louis Lagrange is slanted version of Rolle's theorem. There is a point where the tangent is parallel to chord (secant).

Mean value theorem:

Let f be a function that satisfies the following condition.

- f is continuous on the closed interval $[a, b]$.
- f is differentiable on the open interval (a, b) . Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow f(b) - f(a) = f'(c)(b - a)$$

Note: Geometrically, the theorem guarantees the existence of a tangent line that is parallel to the secant line through the point $(a, f(a))$ and $(b, f(b))$. In terms of rate of change, Mean value theorem implies that there is point c in the open interval (a, b) at which instant rate of change is equal to average rate of change over the interval $[a, b]$.

Illustrative Example

Find all values of c that satisfy the conclusion of the mean value theorem.

a. $f(x) = 4x^2 - 2x + 5, [-1, 1]$

c. $f(x) = \frac{x+2}{x-3}, [4, 6]$

b. $f(x) = x^{\frac{2}{3}} + 1, [0, 1]$

d. $x^3 + x - 1, [0, 2]$

Solution:

a) $f(x) = 4x^2 - 2x + 5$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$, therefore there exist c in $(-1, 1)$

$$\text{such that } f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{11 - 7}{2} = 2 \Rightarrow f'(x) = 8x - 2$$

$$\Rightarrow f'(c) = 8c - 2 = 2 \Rightarrow c = 0 \in [-1, 1].$$

b) $f(x) = x^{\frac{2}{3}} + 1$ is continuous on $[0, 1]$ and differentiable on

$(0, 1)$. Therefore, there is c in $(0, 1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{2 - 1}{1} = 1 \Rightarrow f'(x) = \frac{2}{3} x^{-\frac{1}{3}} = 1$$

$$\Rightarrow x^{-\frac{1}{3}} = \frac{3}{2} \Rightarrow x = \left(\frac{3}{2}\right)^{-3} = \frac{8}{27}$$

$$\therefore c = \frac{8}{27}$$

$$c) f'(c) = \frac{f(6) - f(4)}{6 - 4} = \frac{\frac{8}{3} - 6}{2} = -\frac{5}{3} \Rightarrow f'(x) = \frac{(x-3) - (x+2)}{(x-3)^2} = -\frac{5}{3}$$

$$\Rightarrow (x-3)^2 = 3 \Rightarrow x-3 = \pm\sqrt{3} \Rightarrow x = 3 + \sqrt{3}, x = 3 - \sqrt{3}$$

$\therefore$ Since c must lie in $(4, 6)$, $c = 3 + \sqrt{3}$

d) Exercise left for you. (Answer: $c = \frac{2\sqrt{3}}{3}$)

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8. Let $f(x) = Ax^2 + Bx + C$ where A , B and C are constant with $A \neq 0$. Show that for any interval $[a, b]$ the number c guaranteed by the mean value theorem is the mid-point of $[a, b]$.

$$\text{Solution: } f'(x) = 2Ax + B \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2Ac + B = \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b - a} = \frac{A(b^2 - a^2)}{b - a} + B \left(\frac{b - a}{b - a} \right)$$

$$\Rightarrow 2Ac + B = A(b + a) + B \Rightarrow 2Ac = A(b + a)$$

$$\therefore c = \frac{b + a}{2} \text{ which is mid. pt. of } [a, b].$$

9. The value of c of the mean value theorem if $f(x) = x(x-1)(x-2)$ on $\left[0, \frac{1}{2}\right]$.

A. 1 B. $1 + \frac{\sqrt{21}}{6}$ C. $1 - \frac{\sqrt{21}}{6}$ D. $-1 + \frac{\sqrt{21}}{6}$

Answer: C

Note:

- i) If $f'(x) = g'(x)$ for all $x \in (a, b)$ then there exist a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$. That $f - g$ is constant on (a, b) .
- ii) If $f'(x) = 0$ at each point of an interval I , then f is constant for all x in I .

Absolute Extrema and Range of function

10. Find the range of each of the following function in the indicated interval.

a. $f(x) = 4x^3 - 48x + 3, [-1, 3]$

b. $f(x) = x^3 - 12x + 1, [0, 3]$

c. $f(x) = x^2 - \frac{16}{3}x^3 + 6x^2, (-\infty, \infty)$

Solution: First find the critical number in the given interval.

a. $f(x) = 4x^3 - 48x + 3 \Rightarrow f'(x) = 12x^2 - 48 = 0 \Rightarrow x^2 = \frac{48}{12} = 4$

$\Rightarrow x = \pm 2$, take only $x = 2$. Now, the candidates are $-1, 2$ and 3

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Thus, $f(-1) = -4 + 48 + 3 = 47$ is absolute maximum in $[-1, 3]$.

And $f(2) = 4(8) - 48(2) + 3 = 32 - 96 + 3 = -61$ is absolute minimum. $f(3) = 4(27) - 48(3) + 3 = 108 - 144 + 3 = -31$

$\therefore$ The range of f is $[-61, 47]$.

b. $f(x) = x^3 - 12x + 1 \Rightarrow f'(x) = 3x^2 - 12 \Rightarrow f'(x) = 0$
 $\Rightarrow 3x^2 - 12 = 0 \Rightarrow x = -2, 2 \leftarrow$ critical numbers

$\therefore$ The candidates are $x = 0, 2, 3$

$\Rightarrow f(0) = 1, f(2) = 8 - 24 + 1 = -15$ and $f(3) = -8.$

$\therefore$ The range of f is $[-15, 1]$.

c. $f(x) = x^4 - \frac{16}{3}x^3 + 6x^2 \Rightarrow f'(x) = 4x^3 - 16x^2 + 12x$

$= 4x(x^2 - 4x + 3)$

$\Rightarrow 4x(x-1)(x-3) = 0 \Rightarrow x = 0, 1, 3$ are critical numbers

$\Rightarrow f(0) = 0, f(1) = 1 - \frac{16}{3} + 6 = \frac{5}{3}$ and $f(3) = -9.$

$\therefore$ The range of f is $[-9, \infty).$

Increasing and decreasing function

A function $f(x)$ is said to be:

i) **Increasing** on interval I if for all points x_1 and x_2 in I ,

$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2).$

ii) **Decreasing** on interval I if for all points x_1 and x_2 in I ,

$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2).$

A function $f(x)$ said to be:

i) **Strictly increasing on interval** if it is increasing on the entire interval.

ii) **Strictly decreasing on interval** if it is decreasing on the entire interval.

A function is increasing if as x move to the right, its graph is rising up.

A function is decreasing if as x move to the right, its graph moves down (falling).

- If the derivative $f'(x)$ is positive at x_0 then $f(x)$ is increasing at x_0 .
- If the derivative $f'(x)$ is negative at x_0 , then $f(x)$ is decreasing at x_0 .
- Here, first derivative test used to determine **monotonic** and **local extreme value**.

First derivative test for increasing and decreasing function

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) .

- If $f'(x) > 0$ for all x in (a, b) then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for all x in (a, b) then f is decreasing on $[a, b]$.
- If $f'(x) = 0$, for all x in (a, b) then f is constant.

Illustrative Example

11. Find all the intervals in which f is strictly increasing, decreasing and locate all relative extrema.

a. $f(x) = x^3 - 12x - 5$

b. $f(x) = 17 - 15x + 9x^2 - x^3$

Solution: Here is how we apply the 1st derivative test to find where the function is increasing and decreasing. The critical points of f partitions the x -axis into intervals on which f' is either positive or negative.

a. $f(x) = x^3 - 12x - 5 \Rightarrow f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$
 $\Rightarrow f'(x) = 0 \Rightarrow x = 2, -2$ are critical points so we are going to determine, where $f'(x) > 0$ and $f'(x) < 0$.

From the sign table we conclude that:

$x + 2$	- - -	0	- - -	+	+	+
$x - 2$	- - -	-	- - -	0	- - -	+
$f'(x)$	+	+	+	0	- - -	0

- i) $f'(x) > 0$, on $(-\infty, -2)$ and $(2, \infty)$ and $f'(x) = 0$ only for

$x = -2$ and $x = 2$.

$\therefore f$ is strictly increasing on $(-\infty, -2]$ and $[2, \infty)$.

- ii) $f'(x) < 0$ on $x \in (-2, 2) \Rightarrow f$ is strictly decreasing on $[-2, 2]$.

- iii) Since $f'(x)$ changes the sign from positive to negative at $x = -2 \Rightarrow f$ has local maximum at $x = -2$ and $f'(x)$ change the sign from negative to positive at $x = 2$.

$\Rightarrow f$ has local minimum at $x = 2$.

b. $f'(x) = -15 + 18x - 3x^2 = -3(x^2 - 6x + 5) = -3(x-1)(x-5)$

- i) $f'(x) > 0 \Rightarrow -3(x-1)(x-5) > 0 \Rightarrow x \in (1, 5)$.

$\therefore f$ is strictly increasing on $[1, 5]$.

- ii) $f'(x) < 0 \Rightarrow -3(x-1)(x-5) < 0 \Rightarrow x \in (-\infty, 1)$ and $(5, \infty)$.

$\therefore f$ is strictly decreasing on $(-\infty, 1]$ and $[5, \infty)$.

- iii) f' changes from negative to positive at $x = 1$.

$\therefore f$ has local minimum at $x = 1 \Rightarrow f(1) = 10$.

- iv) f' changes from positive to negative at $x = 5$.

$\therefore f$ has local maximum at $x = 5 \Rightarrow f(5) = 42$

Local extreme values of function on entire Domain

Knowing where the function increases and decreases help us how to test for the nature of local extreme values.

The First Derivative Test for local Extreme

At a critical point $x = c$

- f has a local minimum, if f' changes from negative to positive at c .
- f has a local maximum if f' changes from positive to negative at c .
- f has no local extreme if f' has the same sign on both sides of c . Example: $f(x) = x^3 + 4 \Rightarrow f'(x) = 3x^2 \geq 0$ for all $x \in (-\infty, \infty)$. $\therefore f$ has no local extrema.

Illustrative Example

Find:

- the critical number (If any)
- the interval on which f is increasing or decreasing
- the local maximum and minimum value

a. $f(x) = x^3 - 3x^2 - 24x + 7$

g. $f(x) = (x^2 - 3)^3$

b. $f(x) = 2x^4 - 16x^2$

h. $f(x) = (x - 2)e^{-x}$

c. $f(x) = \frac{4x}{x^2 + 2}$

i. $f(x) = (x^2 - 4)^{\frac{2}{3}}$

d. $f(x) = \frac{x+1}{x-3}$

j. $f(x) = \frac{1}{2}x - \sin x$, on $[0, 2\pi]$

e. $f(x) = (x^2 - 3)e^x$

k. $f(x) = \frac{\ln x}{x}, x > 0$

f. $f(x) = x^2 - 8\ln x, x > 0$

l. $f(x) = \frac{x}{x^2 + 2x + 1}$

Solution: Here

a. $(x^2 - 2x - 8) = 3(x - 4)(x + 2)$

i) $f'(x) = 0 \Rightarrow x = 4$ and $x = -2$

$\therefore 4$ and -2 are critical numbers.

ii) $f'(x) > 0$ on $(-\infty, -2)$ and $(4, \infty)$.

$\therefore f$ is increasing on $(-\infty, -2]$ and $[4, \infty)$.

	$-\infty$	-2	4	∞
$3(x-4)$	---	---	---	0 + + +
$x+2$	---	0 + + +	---	+
$f'(x)$	+++	0	---	0 - - -

$$f'(x) < 0 \text{ on } (-2, 4).$$

$\therefore f$ is decreasing on $[-2, 4]$.

iii) f' changes from positive to negative at $x = 2$.

$\therefore f(-2) = 35$ is local maximum value.

f' changes from negative to positive at $x = 4$.

$\therefore f(4) = -73$ is local minimum.

$$\begin{aligned} \text{b. } f(x) &= 2x^4 - 16x^2 \\ \Rightarrow f'(x) &= 8x^3 - 32x \\ &= 8x(x^2 - 4) \end{aligned}$$

	$-\infty$	-2	0	2	∞
$8x$	---	---	0	+++	+++
$x+2$	---	0	+++	+++	0
$x-2$	---	---	---	---	+++
$f'(x)$	---	0	+++	0	+++

i) $f'(x) = 0 \Rightarrow x = 0, 2$ and -2 are critical numbers.

ii) $f'(x) < 0$ on $(-\infty, -2)$ and $(0, 2)$

$\Rightarrow f$ is decreasing on $(-\infty, -2]$ and $[0, 2]$.

$f'(x) > 0$ on $(-2, 0)$ and $(2, \infty)$.

$\Rightarrow f$ is increasing on $[-2, 0]$ and $[2, \infty)$.

iii) f' changes from negative to positive at $x = -2, 2$

$\Rightarrow f(-2) = -32 = f(2)$ is local minimum.

f' change from positive to negative at $x = 0$

$\Rightarrow f(0) = 0$ is local maximum

$$\text{c. } f(x) = \frac{4x}{x^2 + 2} \Rightarrow f'(x) = \frac{4(x^2 + 2) - 4x(2x)}{(x^2 + 2)^2} = \frac{8 - 4x^2}{(x^2 + 2)^2}$$

$$\Rightarrow f'(x) = 0 \Rightarrow \frac{8 - 4x^2}{(x^2 + 2)^2} \Rightarrow 8 - 4x^2 = 0 \Rightarrow x = \pm\sqrt{2}$$

i) $x = -\sqrt{2}$ and $x = \sqrt{2}$ are critical number.

From the table we conclude that:

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	$-\infty$	$-\sqrt{2}$	$\sqrt{2}$	∞	
$x + \sqrt{2}$	---	0	+++	+++	
$x - \sqrt{2}$	---	---	0	+++	
$f'(x)$	+++	0	---	0	+++

ii) $f'(x) < 0$ when $x < -\sqrt{2}$ and $x > \sqrt{2}$.

Therefore, $f(x)$ is decreasing entirely on $(-\infty, -\sqrt{2}]$ and $[\sqrt{2}, \infty)$.

$f'(x) > 0$ when $-\sqrt{2} < x < \sqrt{2}$, hence $f(x)$ is increasing entirely on $[-\sqrt{2}, \sqrt{2}]$.

iii) The sign of f' changes from negative to positive at $x = -\sqrt{2}$

$$\therefore f(-\sqrt{2}) = \frac{-4\sqrt{2}}{4} = -\sqrt{2} \text{ is local minimum value}$$

The sign of f' changes from positive to negative at $x = \sqrt{2}$

$$\therefore f(\sqrt{2}) = \frac{4\sqrt{2}}{4} = \sqrt{2} \rightarrow \text{local maximum.}$$

d. $f(x) = \frac{x+1}{x-3} \Rightarrow f'(x) = \frac{x-3-(x+1)}{(x-3)^2} = \frac{-4}{(x-3)^2} \neq 0$, for all x .

There is no x -value that makes $f'(x) = 0$.

Therefore does not have critical number.

i) Since $\frac{-4}{(x-3)^2} < 0$, for all x in $(-\infty, \infty)$, f is entirely

decreasing on $(-\infty, \infty)$ except at $x = 3$.

ii) There is no sign change of f' throughout $(-\infty, \infty)$, therefore, no local extreme value.

e. $f(x) = (x^2 - 3)e^x \Rightarrow f'(x) = 2xe^x + (x^2 - 3)e^x = e^x(x^2 + 2x - 3) \Rightarrow f'(x) = 0 \Rightarrow e^x(x-1)(x+3) = 0$

$$\Rightarrow x = 1 \text{ and } x = -3.$$

$\therefore x = 1$ and $x = -3$ are critical number.

From the table we concluded that:

- i) Since $f'(x) > 0$ when $x < -3$ and $x > 1$, f is increasing entirely on $(-\infty, -3]$ and $[1, \infty)$.

$f'(x) < 0$ on $-3 < x < 1 \Rightarrow$
 f is decreasing entirely on $[-3, 1]$.

- ii) The sign of f' changes from positive to negative at $x = -3$

$$\Rightarrow f(-3) = \frac{6}{e^3} \text{ is local}$$

maximum.

The sign of f' changes from negative to positive at $x = 1$

$$\Rightarrow f(1) = -2e \text{ is local minimum}$$

	$-\infty$	-3	1	∞
e^x	+++	0	+++	+++
$x-1$	---	---	0	+++
$x+3$	---	0	+++	0
$f'(x)$	+++	---	---	+++

f. $f(x) = x^2 - 8 \ln x \Rightarrow f'(x) = 2x - \frac{8}{x} = \frac{2x^2 - 8}{x}, x > 0$

$$\Rightarrow f'(x) = 0 \Rightarrow \frac{2x^2 - 8}{x} = 0 \Rightarrow 2x^2 - 8 = 0 \Rightarrow x = \pm 2$$

$\therefore x = 2$ is the only critical number.

	$-\infty$	0	2	∞
x		0	+++	+++
$2x^2 + 8$			---	0
$f'(x)$			---	0

From the table we conclude that:

- i) $f'(x) < 0$ when $0 < x < 2$
 $\Rightarrow f$ is decreasing on $(0, 2]$

$f'(x) > 0$ when $2 < x < \infty \Rightarrow f$ is increasing on $[2, \infty)$.

- ii) The sign of f' changes from negative to positive at $x = 2$
 $\Rightarrow f(2) = 4 - 8 \ln 2$ is local minimum.

g. $f(x) = (x^2 - 3)^3$
 $\Rightarrow f'(x) = 3(x^2 - 3)^2(2x)$
 $= 6x(x^2 - 3)^2$

$$f'(x) = 0 \Rightarrow 6x(x^2 - 3)^2 = 0$$

$\therefore x = -\sqrt{3}, 0, \sqrt{3}$ are critical numbers.

	$-\infty$	$-\sqrt{3}$	0	$\sqrt{3}$	∞
$6x$	---	---	0	+++	+++
$(x^2 - 3)^2$	+++	0	+++	+++	0
$f'(x)$	+++	0	---	0	+++

From the table we conclude that:

i) $f'(x) < 0$ for all x in $(-\infty, 0) \Rightarrow f$ is decreasing on $(-\infty, 0]$.

$f'(x) > 0$ for all x in $(0, \infty) \Rightarrow f$ is increasing on $[0, \infty)$.

ii) Even if $x = -\sqrt{3}$ and $x = \sqrt{3}$ are critical numbers,

$x = -\sqrt{3}$ and $x = \sqrt{3}$ are not local extreme value because f' has no sign change at $x = -\sqrt{3}$ and $x = \sqrt{3}$. However, the sign changes from negative to positive at $x = 0$

$\Rightarrow f(0) = -27$ is local minimum value.

h) $f(x) = (x-2)e^{-x} \Rightarrow f'(x) = e^{-x} + -(x-2)e^{-x} = e^{-x}(3-x)$

$f'(x) = 0 \Rightarrow e^{-x}(3-x) = 0 \Rightarrow x = 3$ is critical number.

i) $f'(x) > 0 \Rightarrow e^{-x}(3-x) > 0 \Rightarrow 3-x > 0 \Rightarrow x < 3 \Rightarrow f$ is entirely increasing on $(-\infty, 3]$.

$f'(x) < 0 \Rightarrow e^{-x}(3-x) < 0 \Rightarrow 3-x < 0 \Rightarrow x > 3 \Rightarrow f$ is entirely decreasing on $[3, \infty)$.

ii) The sign of f' change from positive to negative at $x = 3$.

$\Rightarrow f(3) = (3-2)e^{-3} = e^{-3} \leftarrow$ local maximum value.

j) $f(x) = (x^2 - 4)^{\frac{2}{3}} \Rightarrow f'(x) = \frac{2}{3}(x^2 - 4)^{-\frac{1}{3}}(2x) = \frac{4x}{3\sqrt[3]{x^2 - 4}}$

$\Rightarrow x = 0, -2$ and 2 are critical numbers

$$f(x) = (x^2 - 4)^{\frac{2}{3}}$$

	$-\infty$	-2	0	2	∞
$\frac{4}{3}x$	- - -	- - -	0 + + +	+ + +	+ + +
$x+2$	- - -	0 + + +	+ + +	+ + +	+ + +
$x-2$	- - -	- - -	- - -	0 + + +	+ + +
$\sqrt[3]{x^2-4}$	+ + +	0 - - -	- - -	0 + + +	+ + +
$f'(x)$	- - -	0 + + +	- - -	0 + + +	+ + +

From the table we conclude that:

i) $f'(x) < 0$ when $x < -2$ and $0 < x < 2 \Rightarrow f$ is decreasing on $(-\infty, -2]$ and $[0, 2]$.

$f'(x) > 0$ when $-2 < x < 0$ and $x > 2 \Rightarrow f$ is increasing on $[-2, 0]$ and $[2, \infty)$.

ii) $f(-2) = 0$ and $f(2) = 0$ are local minimum values.

$f(0) = \sqrt[3]{16}$ is local maximum value.

k) $f(x) = \frac{1}{2}x - \sin x$

	0	$\frac{\pi}{3}$	$\frac{5\pi}{3}$	2π
$\frac{1}{2} - \cos x$	- - -	0 + + +	0 - - -	- - -
$f'(x)$	- - -	0 + + +	0 - - -	- - -

$$\Rightarrow f'(x) = \frac{1}{2} - \cos x$$

$$\therefore f'(x) = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3} \text{ are critical number,}$$

$$\text{i) } f'(x) < 0 \text{ when } 0 < x < \frac{\pi}{3} \text{ and } \frac{5\pi}{3} < x < 2\pi \Rightarrow f \text{ is decreasing}$$

$$\text{on } \left[0, \frac{\pi}{3}\right] \text{ and } \left[\frac{5\pi}{3}, 2\pi\right] \quad f'(x) > 0 \text{ when } \frac{\pi}{3} < x < \frac{5\pi}{3}$$

$$\therefore f \text{ is increasing on } \left[\frac{\pi}{3}, \frac{5\pi}{3}\right].$$

$$\text{ii) The sign of } f' \text{ change from negative to positive at } x = \frac{\pi}{3}$$

$$(\text{decreasing then increasing}) \Rightarrow f\left(\frac{\pi}{3}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{2} \text{ is local minimum value. The sign of } f' \text{ change from positive to}$$

$$\text{negative at } x = \frac{5\pi}{3} \text{ (increasing then decreasing)}$$

$$\Rightarrow f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{6} - \frac{\sqrt{3}}{2} \leftarrow \text{local maximum value.}$$

$$\text{k. } f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{1 - \ln x}{x^2}, \quad f'(x) = 0 \Rightarrow \frac{1 - \ln x}{x^2} = 0$$

$$\Rightarrow \ln x = 1 \Rightarrow x = e \Rightarrow x = e \text{ is critical number}$$

From the table we concluded that:

$$\text{i) } f'(x) > 0 \text{ when } 0 < x < e \Rightarrow f \text{ is increasing on } (0, e].$$

$$f'(x) < 0 \text{ when } x > e \Rightarrow f \text{ is decreasing on } [e, \infty).$$

	0	e	∞
$1 - \ln x$	+++	0	---
x^2	---	+++	+++
$f'(x)$	+++	0	---

- ii) The sign of f' change from positive to negative at $x = e$
(increasing then decreasing) $\Rightarrow f(e) = \frac{1}{e}$ is local max.

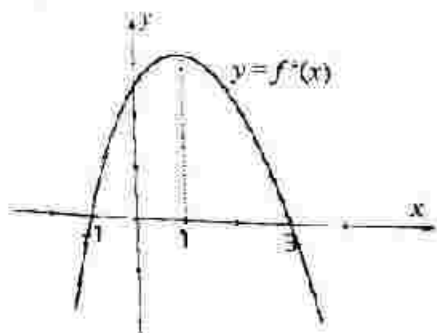
1. $f(x) = \frac{x}{x^2 + 2x + 1}$ is exercise left for you.

Determining monotonic and extreme value of function using graphs of derivative of function

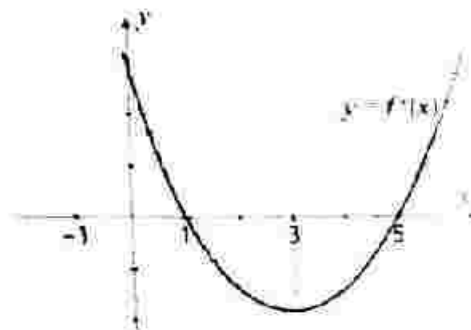
Illustrative Example

13. Suppose $f(x)$ is differentiable at all $x \in (-\infty, \infty)$. Then from the graph of $y = f'(x)$ given below, find:

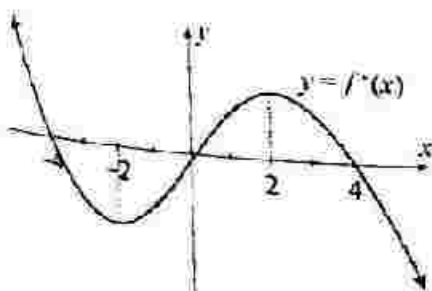
- The intervals on which the function f is increasing.
- The intervals on which the function f is decreasing.
- The extreme value.



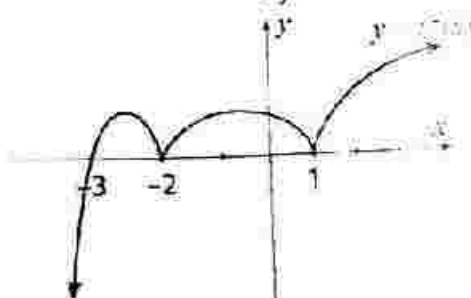
a)



b)



c)



d)

Solution: If the graph lies above the x -axis $f' > 0$. If the graph lies below the x -axis $f' < 0$.

a) i) since $x \in (-1, 3) \Rightarrow f'(x) > 0 \Rightarrow f$ is increasing on $[-1, 3]$.

ii) since $x \in (-\infty, -1)$ and $x \in (3, \infty) \Rightarrow f'(x) < 0$.
 $\therefore f$ is decreasing on $(-\infty, -1]$ and $[3, \infty)$.

iii) Since $f'(-1) = 0$ and $f'(3) = 0$, $x = -1$, and $x = 3$ are critical numbers.

The sign of $f'(x)$ change from negative to positive at $x = -1$
 $\Rightarrow (-1, f(-1))$ is local minimum value.

Sign of $f'(x)$ change from positive to negative at $x = 3$

$\Rightarrow (3, f(3))$ is local maximum value.

Note: you can see that $f''(1) = 0$.

b) Using sign chart of $f'(x)$

	$-\infty$	1	5	∞
$y = f'(x)$	+++	---	+++	
$y = f(x)$	Increase	Decrease	Increase	

i) $f(x)$ is increasing on $(-\infty, 1]$ and $[5, \infty)$.

ii) $f(x)$ is decreasing on $[1, 5]$.

iii) $f'(1) = 0$ and $f'(5) = 0$

The sign of $f'(x)$ change from positive to negative at $x = 1$

$\Rightarrow f(x)$ attains is local maximum value at $x = 1$.

Sign of $f'(x)$ change from negative to positive at $x = 5$

$\Rightarrow f(x)$ attains is local minimum value at $x = 5$.

Note: We can observe that $f''(3) = 0$.

c) Using sign chart of $f'(x)$

	$-\infty$	-4	0	4	∞
$y = f'(x)$	---	---	---	+++	---
$y = f(x)$	Decrease	Decrease	Increase	Decrease	

i) $f'(x) > 0$ on $(-\infty, -4)$ and $(0, 4)$

$\Rightarrow f(x)$ is increasing on $(-\infty, -4]$

ii) $f'(x) < 0$ on $(-4, 0)$ and $(4, \infty)$

$\Rightarrow f(x)$ is decreasing on $[-4, 0]$ and $[4, \infty)$.

iii) $x = -4, 0$ and 4 are local extreme value

$\Rightarrow (-4, f(-4))$ and $(4, f(4))$ are local maximum and $(0, f(0))$ is local minimum value.

d) Using sign chart:

	$-\infty$	-3	-2	1	∞
$y = f'(x)$	---	+++	+++	+++	
$y = f(x)$	Decrease	Increase	Increase	Increase	

i) $f'(x) > 0$ on $(-3, -2)$, $(-2, 1)$ and $(1, \infty) \Rightarrow f(x)$ is increasing on $[-3, \infty)$.

ii) $f'(x) < 0$ on $(-\infty, -3) \Rightarrow f(x)$ is decreasing on $(-\infty, -3]$.

iii) $f'(-3) = 0$, $f'(-2) = 0$, and $f'(1) = 0 \Rightarrow x = -3, -2$ and 1 are critical numbers. But only at $x = -3$ is local extreme value because no sign change at $x = -2$ and $x = 1$.

$\therefore f(-3) = -20 \leftarrow$ local minimum value.

Which of the following is local minimum point of the function

14. $f(x) = -3x^4 + 4x^3 + 12x^2 + 1 \dots$ (UEE)

A. $(-2, -33)$ B. $(-1, 4)$ C. $(0, 1)$ D. $(2, 33)$

Solution: $f'(x) = -12x^3 + 12x^2 + 24x = -12x(x^2 - x - 2)$

$$f'(x) = 0 \Rightarrow -12x(x-2)(x+1) = 0.$$

$\therefore x = 0, 2$ and -1 are critical numbers $\Rightarrow f''(x) = -36x^2 + 24x + 24$

i) $f'(0) = 0$ and $f''(0) = 24 > 0$ $(0, 1)$ is local minimum value.

ii) $f'(-1) = 0$ and $f''(-1) = -36 < 0 \Rightarrow (-1, 6)$ is local maximum.

iii) $f'(2) = 0$ and $f''(2) = -72 < 0 \Rightarrow (2, 33)$ is local maximum

Answer: C

The second derivative test for local extreme

i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .

ii) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .

Illustrative Example

15. Using the second derivative test, find the extreme value of:

a. $f(x) = 4x^3 + 21x^2 + 36x + 7$

b. $f(x) = -2x^3 - 6x^2 - 5$

Solution:

a. $f'(x) = 12x^2 + 42x + 36 = 6(2x+3)(x+2)$ and $f''(x) = 24x + 42$

$$f'(x) = 0 \Rightarrow x = -\frac{3}{2} \text{ and } x = -2.$$

$f'\left(-\frac{3}{2}\right) = 0$ and $f''\left(-\frac{3}{2}\right) = 78 > 0 \Rightarrow \left(-\frac{3}{2}, f\left(-\frac{3}{2}\right)\right)$ is local minimum and $f'(-2) = 0$ and $f''(-2) = -6 < 0 \Rightarrow (-2, f(-2))$ is local maximum.

b. Exercise left for you.

16. Let $f(x) = x^4 - 6x^2 + 1$ on $[-2, 3]$ be given, then which of the following a local maximum value of $f \dots$ (UEE)

A. $f(0)$ B. $f(\sqrt{3})$ C. $f(3)$ D. $f(-\sqrt{3})$

Solution: $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$

$$f'(x) = 0 \Rightarrow 4x(x^2 - 3) = 0 \Rightarrow x = 0, \sqrt{3} \text{ and } -\sqrt{3}.$$

Thus, $x = -2, -\sqrt{3}, 0, \sqrt{3}$ and 3 are candidate for maximum.
 $f(-2) = -7, f(3) = 28$ is local maximum,
 $f(0) = 1, f(\sqrt{3}) = f(-\sqrt{3}) = -8$ is local minimum.

Answer: C

Note: The range of f on $[-2, 3]$ is $[-8, 28]$

The range of f on $(-\infty, \infty)$ is $[-8, \infty)$.

17. The maximum value of $f(x) = x^2 - 6x$ on the interval $[-4, 4]$.
 A. $f(-4)$ B. $f(3)$ C. $f(-4)$ D. $f(1)$

Solution: $f'(x) = 2x - 6$

$f'(x) = 0 \Rightarrow 2x - 6 = 0 \Rightarrow x = 3$. Thus, $x = 3$ is critical number. $x = 3, -4, 4$ are candidates for maximum

$$\Rightarrow f(-4) = 16 + 24 = 40, f(3) = -9 \text{ and } f(4) = 16 - 24 = -8.$$

Answer: A

18. Find the value of k , for which the function will have extreme value at the indicated point.

a. $f(x) = \frac{x+3}{x^2+k}$, at $x=2$ b. $f(x) = \frac{x^2-2x+6k}{x-2}$, at $x=-1$

Solution: Recall relative extreme value occur at critical number.

a. $f(x) = \frac{x+3}{x^2+k} \Rightarrow f'(x) = \frac{x^2+k - (2x)(x+3)}{(x^2+k)^2} = \frac{-x^2-6x+k}{(x^2+k)^2}$

since at $x = 2$ is extreme value of function, so that $f'(2) = 0$

$$\Rightarrow f'(2) = \frac{-4-12+k}{(4+k)^2} = 0 \Rightarrow 16+k=0 \Rightarrow k = -16$$

b. $f(x) = \frac{x^2-2x+6k}{x-2} \Rightarrow f'(x) = \frac{(2x-2)(x-2) - (x^2-2x+6k)}{(x-2)^2}$

$$\Rightarrow f'(x) = \frac{2x^2-6x+4-x^2+2x-6k}{(x-2)^2} = \frac{x^2-4x+4-6k}{(x-2)^2}$$

Since at $x = -1$ is extreme value of f , so that $f'(-1) = 0$.

$$\Rightarrow f'(-1) = \frac{1+4+4-6k}{(-3)^2} = 0 \Rightarrow 9-6k=0 \Rightarrow k = \frac{3}{2}$$

Applications of Differential Calculus

For what values of k so that $f(x) = \frac{x}{x^2 + k}$ will have extreme values at $x = 2 \dots$ (UEF)

- A. 0 B. 2 C. 4 D. 27

Solution: $f'(x) = \frac{x^2 + k - (2x)x}{(x^2 + k)^2} = \frac{k - x^2}{(x^2 + k)^2}$ since at $x = 2$ is an extreme value of f so that $f'(2) = 0$

$$\Rightarrow f'(2) = \frac{k - 4}{4 + k} = 0 \Rightarrow k - 4 = 0 \Rightarrow k = 4,$$

Answer: C

Concavity and inflection points

In the previous section we used to first derivative test help us to determine monotonicity intervals, local maximum value and local minimum value. Now, here that second derivative test is also important to determine on what interval the function is bending up (concave up) or bending down (concave down ward). Function is:

- Concave up then the graph of f is bending upward
- Concave down when the graph of f is bending down ward

Second derivative test for concavity

Let f be a function whose second derivative exists on an open interval I , then:

- If $f''(x) > 0$ for all x in I , then the graph of f is concave upward in I .
- If $f''(x) < 0$ for all x in I , then the graph of f is concave down ward in I .

Point of inflection

The point where the graph of f changes from concave up to concave down or concave down to concave up is called point of inflection. If $(c, f(c))$ is point of inflection of the graph of f , then either $f''(c) = 0$ or $f''(c)$ does not exist.

Illustrative Example

20. Find:

- | | |
|---|---|
| i) concavity interval | ii) inflection point |
| iii) local extreme value for each of the following. | |
| a) $f(x) = x^4 - 4x^3 + 5$ | d) $f(x) = (\ln x)^2$ |
| b) $f(x) = 4x^3 - 12x^2 + 9$ | e) $f(x) = 5 + \sin x$, on $(0, 2\pi]$ |

$$c) \quad f(x) = \frac{4}{x^2 + 3}$$

Solution: Use second derivative test

$$a) \quad f(x) = x^4 - 4x^3 \Rightarrow f'(x) = 4x^3 - 12x^2 \Rightarrow f''(x) = 12x^2 - 24x$$

$$f'(x) = 0 \Rightarrow 4x^2(x - 3) = 0 \Rightarrow x = 0 \text{ and } 3 \text{ are critical numbers}$$

$$f''(x) = 0 \Rightarrow 12x(x - 2) = 0 \Rightarrow x = 0 \text{ and } 2 \text{ is inflection point}$$

Using sign chart

	$-\infty$	0	2	∞
$12x$	---	0	+++	+++
$x - 2$	---	---	0	+++
$y = f''(x)$	+++	0	---	+++

From the table (2^{nd} derivative test)

i) The graph is concave up on $(-\infty, 0)$ and $(2, \infty)$. The graph is concave down on $(0, 2)$.

ii) Since $f''(0) = 0$ and $f''(2) = 0$, therefore $(0, f(0))$ and $(2, f(2))$ are inflection point.

iii) Since $x = 0$ and $x = 3$ are critical number so that from 2^{nd} derivative for local extreme value

$$f'(0) = 0 \text{ and } f''(0) = 0. \text{ Therefore, not extreme value}$$

$$f'(3) = 0 \text{ and } f''(3) = 36 > 0. \text{ Therefore, has local minimum.}$$

$$\therefore \text{The only local extreme value is } (3, f(3)) = (3, -12)$$

$$b. \quad f(x) = 4x^3 - 12x^2 + 9 \Rightarrow f'(x) = 12x^2 - 24x \Rightarrow f''(x) = 24x - 24$$

$$f'(x) = 0 \Rightarrow 12x(x - 2) = 0 \Rightarrow x = 0 \text{ and } 2 \text{ are critical numbers}$$

$$f''(x) = 0 \Rightarrow 24x - 24 = 0 \Rightarrow x = 1 \text{ is inflection point}$$

The graph of f has:

i) **Concave up** when $f''(x) > 0 \Rightarrow 24x - 24 > 0 \Rightarrow x > 1$

ii) **Concave down** when $f''(x) < 0 \Rightarrow 24x - 24 < 0 \Rightarrow x < 1$

iii) **local extreme if**

$$1) \quad f'(0) = 0 \text{ and } f''(0) = -24 < 0 \leftarrow \text{local max at } 0.$$

$$\therefore (0, f(0)) = (0, 9) \leftarrow \text{local maximum value}$$

$$2) \quad f'(2) = 0 \text{ and } f''(2) = 48 - 24 = 24 > 0 \rightarrow \text{local min}$$

$$\therefore (2, f(2)) = (2, -7) \leftarrow \text{local minimum value}$$

Applications of Differential Calculus

$$c. f(x) = \frac{4}{x^2 + 3} \Rightarrow f'(x) = \frac{-8x}{(x^2 + 3)^2}$$

$$\Rightarrow f''(x) = \frac{-8(x^2 + 3)^2 - (2(x^2 + 3)(2x)(8x))}{(x^2 + 3)^4}$$

$$\Rightarrow f''(x) = \frac{-8(x^2 + 3) + 32x^2}{(x^2 + 3)^3} = \frac{24x^2 - 24}{(x^2 + 3)^3} = \frac{24(x^2 - 1)}{(x^2 + 3)^3}$$

$$f'(x) = 0 \Rightarrow -8x = 0 \Rightarrow x = 0 \leftarrow \text{critical number}$$

$$f''(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1 \text{ and } x = -1 \leftarrow \text{inflection}$$

Now, using sign chart table

for $f''(x)$

	$-\infty$	-1	1	∞
$24(x^2 + 3)^3$	+++	+++	+++	+++
$x - 1$	---	---	0	+++
$x + 1$	---	0	+++	---
$y = f''(x)$	+++	0	---	0

i) f is concave up on $(-\infty, -1)$ and $(1, \infty)$.

f is concave down on $(-1, 1)$

ii) Point of inflection at $x = -1$ and $x = 1$ $(-1, f(-1)) = (-1, 1)$ and $(1, 1)$.

iii) $f'(0) = 0$ and $f''(0) = \frac{-8}{9} < 0$ Therefore, local max at $x = 0$.

$$d. f(x) = (\ln x)^2 \Rightarrow f'(x) = \frac{2 \ln x}{x} \Rightarrow f''(x) = \frac{2 - 2 \ln x}{x^2}$$

$$\Rightarrow f'(x) = 0 \Rightarrow \frac{2 \ln x}{x} = 0 \Rightarrow x = 1 \leftarrow \text{critical number}$$

$$\Rightarrow f''(x) = 0 \Rightarrow \frac{2 - 2 \ln x}{x^2} = 0 \Rightarrow 2 - 2 \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$$

Using sign chart

Then, from the table we concluded that the graph of f is:

i) Concave up on $(0, e)$ and concave down on (e, ∞)

ii) At $x = e$ is inflection point $(e, 1)$

iii) Since $f'(1) = 0$ and $f''(1) = 2 > 0 \leftarrow \text{local min at } x = 1$.

$\therefore (1, f(1)) = (1, 0)$ is local minimum value.

$$e. f(x) = 5 + \sin x \Rightarrow f'(x) = \cos x \Rightarrow f''(x) = -\sin x$$

	0	e	∞
$2(1 - \ln x)$	+++	0	---
x^2	+++	+++	+++
$y = f''(x)$	+++	0	---

Graph of $f(x)$ Concave up Concave down

$$i) f'(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \leftarrow \text{critical number}$$

$$ii) f''(x) = 0 \Rightarrow -\sin x = 0 \Rightarrow x = 0, \pi, 2\pi \leftarrow \text{inflection point}$$

From the table we conclude that the graph of f is:

- i) Concave up on $(\pi, 2\pi)$ and concave down on $(0, \pi)$

	0	π	2π
$-\sin x$	---	0	+++
$f''(x)$	---	0	+++

- ii) At $x = \pi$ is inflection point $(\pi, 5)$

$$iii) f'\left(\frac{\pi}{2}\right) = 0 \text{ and } f''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$$

$$\therefore \text{local maximum at } x = \frac{\pi}{2}$$

$$f'\left(\frac{3\pi}{2}\right) = 0 \text{ and } f''\left(\frac{3\pi}{2}\right) = -\sin\left(\frac{3\pi}{2}\right) = 1$$

$$\therefore \text{local minimum at } x = \frac{3\pi}{2}$$

$$\text{That is } \left(\frac{\pi}{2}, f\left(\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2}, 6\right) \text{ is local maximum value}$$

$$\left(\frac{3\pi}{2}, f\left(\frac{3\pi}{2}\right)\right) = \left(\frac{3\pi}{2}, 4\right) \text{ is local minimum value}$$

$$f. \quad f(x) = x^4 e^{2x} \Rightarrow f'(x) = 4x^3 e^{2x} + x^4 e^{2x}(2) = e^{2x} x^3 (4 + 2x) \\ \Rightarrow f''(x) = 2e^{2x} x^3 (4 + 2x) + 3x^2 e^{2x} (4 + 2x) + e^{2x} x^3 (2) \\ = 4x^2 e^{2x} (x^2 + 4x + 3)$$

$$f'(x) = 0 \text{ when } e^{2x} x^3 (4 + 2x) = 0 \Rightarrow x = 0, -2 \text{ are critical numbers}$$

$$\Rightarrow f''(x) = 0 \text{ when } 4x^2 e^{2x} (x^2 + 4x + 3) = 0$$

$$\Rightarrow 4x^2 e^{2x} (x + 3)(x + 1) = 0$$

$$\Rightarrow x = 0, -3, -1 \text{ are inflection points}$$

Using sign chart from 2nd derivative:

From the table we see that:

- i) $f''(x) > 0$ on $(-\infty, -3)$, $(-1, 0)$ and $(0, \infty)$

	$-\infty$	-3	-1	0	∞
$e^{2x} 4x^2$	+++	+++	+++	0	+++
$x + 3$	---	0	+++	+++	+++
$x + 1$	---	---	0	+++	+++

∴ The graph of $f(x)$ is concave up on $(-\infty, -3)$, $(-1, 0)$ and $(0, \infty)$ similarly $f''(x) < 0$ on $(-3, -1) \Rightarrow$ the graph of $f(x)$ is concave down on $(-3, -1)$.

ii) At $x = -3$ and $x = -1$ are inflection point of the graph of function. But at $x = 0$ the graph has no inflection because no sign change at 0.

iii) $f'(-2) = 0$ and $f''(-2) = -64e^{-4} < 0 \Rightarrow (-2, 16e^{-4})$ is local maximum value

Note: For $a, b \in \mathbb{R}$

i) $f(x) = ax + b \Rightarrow f'(x) = a \Rightarrow f''(x) = 0$ (concavity doesn't exist)

ii) $f(x) = x^4 + a \Rightarrow f'(x) = 4x^3 \Rightarrow f''(x) = 12x^2 > 0$ for all $x \neq 0$
∴ It is concave up on $(-\infty, \infty)$.

∴ No inflection point even if $f''(0) = 0$ because no sign change.

11. The graph of $f(x) = x^4 - 6x^2 + 1$ has an inflection point at (UEE)

A. $(-1, 1)$ B. $(1, -4)$ C. $(\sqrt{3}, 1)$ D. $(0, 1)$

Solution:

$$f(x) = x^4 - 6x^2 + 1 \Rightarrow f'(x) = 4x^3 - 12x \Rightarrow f''(x) = 12x^2 - 12$$

$$\text{Then } f''(x) = 0 \Rightarrow 12(x^2 - 1) = 0 \Rightarrow x = -1, 1$$

Answer: B

$$\Rightarrow (-1, f(1)) = (-1, -4) \text{ and } (1, f(1)) = (1, -4) \text{ are inflection.}$$

18. Which of the following is true about $f(x) = x^3 - 3x^2 + 3x + 1$.

A. f is strictly decreasing on $(-\infty, \infty)$

B. f is strictly increasing on $(-\infty, \infty)$

C. f has local extreme value at $x = 1$

D. $(1, 0)$ is the inflection point of the graph

Solution:

$$f(x) = x^3 - 3x^2 + 3x + 1 \Rightarrow f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2 > 0$$

$$\Rightarrow f'(x) > 0 \text{ for all } x, \text{ except at } x = 1.$$

∴ f is strictly increasing on $(-\infty, \infty)$. Even if $f'(1) = 0$, is critical number but $x = 1$ is not local extreme value because f' has no sign change at 1 similarly $\Rightarrow f''(x) = 6x - 6 \Rightarrow f''(x) = 0$ when $6x - 6 = 0 \Rightarrow x = 1$. Now, $f''(x) > 0$ when $x > 1$ and $f''(x) < 0$ when $x < 1$.

∴ $(1, f(1)) = (1, 2)$ is inflection point.

Answer: B

Minimization and Maximization problem

One of the most common application of calculus involves the determination of minimum and maximum value. Consider the following terms you hear frequently. Some of them are: maximize, profit, minimize cost, least time, least size, maximize strength, maximize distance, ... etc.

I. Maximizing and minimizing of product and sum of number**Illustrative Example****Finding two non-negative numbers**

19. a. Find the two positive number whose the sum is 15 and the product is maximum.

Solution: $x \geq 0, y \geq 0$, and $x + y = 15$, with product (x, y) max.

Sum: $x + y = 15 \Rightarrow y = 15 - x$, product $(P) = xy \leftarrow$ to be max.

Product: $P(x) = x(15 - x) = 15x - x^2$

$$\Rightarrow P'(x) = 15 - 2x, 0 \leq x \leq 15 \Rightarrow P'(x) = 0 \Rightarrow 15 - 2x = 0$$

$$\Rightarrow x = \frac{15}{2}, y = 15 - \frac{15}{2} = \frac{15}{2}$$

$\therefore$ The two numbers are $x = \frac{15}{2}$ and $y = \frac{15}{2}$ whose product is maximum.

- b. Find two positive number, whose sum is 18 and the product is maximum.

Solution: Exercise left for you. (Answer, 9 and 9)

- c. Find two positive number whose product is 192 and the sum is minimum.

Solution:

$x \geq 0, x \geq 0, xy = 192$, and $x + y$ is to be minimum. $xy =$

$$192 \Rightarrow y = \frac{192}{x} \text{ put in } y \text{ for } x + y \Rightarrow \text{sum } (S(x)) = x + \frac{192}{x}$$

$$\Rightarrow S'(x) = 1 - \frac{192}{x^2}, S'(x) = 0 \Rightarrow x^2 = 192$$

$$\Rightarrow x = 8\sqrt{3}, \text{ so that } y = \frac{192}{x} = \frac{192}{8\sqrt{3}} = \frac{8\sqrt{3}}{3}$$

- d. Find two number whose product is 192 the sum of the second number and three times the first is minimum

Solution: Let x be the first and y be the second number.

Then $xy = 192$ and $y + 3x$ is to be minimized $\Rightarrow y = \frac{192}{x}$ put in y

$$\text{for } y + 3x \Rightarrow f(x) = \frac{192}{x} + 3x \Rightarrow f'(x) = 3 - \frac{192}{x^2}$$

$\Rightarrow f'(x) = 0$ if $x = 8$, then $y = 24$.

e. What positive number x minimize the sum of x and twice its reciprocal?

Solution:

$$f(x) = x + \frac{2}{x} \Rightarrow f'(x) = 1 - \frac{2}{x^2} \Rightarrow f'(x) = 0 \Rightarrow x = \sqrt{2}$$

$$\therefore \text{The minimum values is } x + \frac{2}{x} = \sqrt{2} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

20. Maximizing area of rectangle and minimize perimeter

a. Find the length (x) and width (y) of a rectangle with perimeter 100 m that maximize area.

Solution:

$$\text{Perimeter (P)} = 2x + 2y = 100 \Rightarrow x = y = 50 \Rightarrow y = 50$$

$$\text{Area (A)} = xy \Rightarrow A(x) = x(50 - x) = 50x - x^2 \Rightarrow A'(x) = 50 - 2x$$

$$\Rightarrow A'(x) = 0 \text{ when } x = 25, \text{ so that } y = 25.$$

$\therefore$ length $x = 25$ and width $w = 25$ that gives the maximum area.

b. Find the length (x) and width (y) of rectangle with area 64 m² that minimize the perimeter.

Solution:

$$\text{Area of rectangle} = xy = 64 \Rightarrow y = \frac{64}{x} \quad \text{Perimeter } 2x + 2y = 100$$

$$\text{minimized } P(x) = 2x + 2\left(\frac{64}{x}\right) = 2x + \frac{128}{x}$$

$$\Rightarrow P'(x) = 2 - \frac{128}{x^2} \Rightarrow P'(x) = 0 \text{ when } 2 - \frac{128}{x^2} = 0 \Rightarrow 2x^2 = 128$$

$$\Rightarrow x^2 = 64 \Rightarrow x = 8, \text{ then } y = \frac{64}{x} = \frac{64}{8} = 8.$$

$\therefore$ Length $x = 8$ and width $y = 8$ that gives the minimum perimeter.

21. **Finding the point closest to the graph of f .**

Find the point on the graph of the function that are closest to the given point.

a. $f(x) = x^2 + 1, (0, 4)$ d. $f(x) = x^2 + 2x, (-1, 0)$

b. $f(x) = 4 - x^2, (0, 2)$ e. $f(x) = 2x - 4, (1, 3)$

c. $f(x) = x^2, \left(2, \frac{1}{2}\right)$ f. $f(x) = (x + 1)^2, (5, 3)$

Solution: Let d be the distance between the point $(0, 4)$ and a point (x, y) on the graph $y = x^2 + 1$ is given by

$$d = \sqrt{(x - 0)^2 + (y - 4)^2} \text{ quantity to be minimized.}$$

$$\Rightarrow d(x) = \sqrt{x^2 + (x^2 + 1 - 4)^2} = \sqrt{x^4 - 5x^2 + 9} \text{ put } x^2 + 1 \text{ in } y.$$

$$\text{Let } g(x) = x^4 - 5x^2 + 9 \Rightarrow g'(x) = 4x^3 - 10x \Rightarrow g'(x) = 0 \text{ when } 2x$$

$$(2x^2 - 5) = 0 \Rightarrow x = 0, \sqrt{\frac{5}{2}} \text{ and } -\sqrt{\frac{5}{2}} \text{ are critical number. To}$$

identify, use second derivative test

$$\Rightarrow g''(x) = 12x^2 - 10 \Rightarrow g''(0) = -10 < 0 \text{ (maximum)}$$

$$\text{but } g''\left(\sqrt{\frac{5}{2}}\right) > 0 \text{ and } g''\left(-\sqrt{\frac{5}{2}}\right) > 0 \text{ minimum.}$$

$$\therefore \text{The point } \left(-\sqrt{\frac{5}{2}}, \frac{7}{2}\right) \text{ and } \left(\sqrt{\frac{5}{2}}, \frac{7}{2}\right) \text{ are closes to } (0, 4).$$

b. Let d be the distance between the point $(0, 2)$ and a point (x, y) on

$$y = 4 - x^2, d = \sqrt{(x - 0)^2 + (y - 2)^2}$$

$$\Rightarrow d(x) = \sqrt{x^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}$$

$$\text{Let } g(x) = x^4 - 3x^2 + 4 \text{ to be minimized}$$

$$\Rightarrow g'(x) = 4x^3 - 6x = 2x(2x^2 - 3) = 0 \Rightarrow x = 0, -\sqrt{\frac{3}{2}} \text{ and } \sqrt{\frac{3}{2}}$$

$$\Rightarrow g''(x) = 12x^2 - 6 \Rightarrow g''(0) = -6, \text{ so, } x = 0 \text{ yield relative max.}$$

$\Rightarrow g''\left(-\sqrt{\frac{3}{2}}\right) = 12$, so that $x = -\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$ yield minimum.

The point $\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ and $\left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ are point on the graph closest to $(0, 2)$.

e. Let d be the distance between the point $\left(2, \frac{1}{2}\right)$ and point (x, y) on $y = x^2$.

$$\Rightarrow d = \sqrt{(x-2)^2 + \left(y - \frac{1}{2}\right)^2}$$

$$\Rightarrow d(x) = \sqrt{(x-2)^2 + \left(x^2 - \frac{1}{2}\right)^2} = \sqrt{x^4 - 4x + \frac{17}{4}}$$

$$\text{Let } g(x) = x^4 - 4x + \frac{17}{4} \Rightarrow g'(x) = 4x^3 - 4 \Rightarrow g'(x) = 0 \Rightarrow x = 1$$

$$\Rightarrow g''(x) = 12x^3 \Rightarrow g''(1) = 12 > 0, \text{ thus } x = 1 \text{ gives min. dist.}$$

The point $(1, 1)$ is closest to $\left(2, \frac{1}{2}\right)$.

d. Exercise left for you.

e. Let d is distance b/n the point $(1, 3)$ and a point (x, y) on

$$y = 2x - 4. \Rightarrow d = \sqrt{(x-1)^2 + (y-3)^2}$$

$$\Rightarrow d(x) = \sqrt{(x-1)^2 + (2x-4-3)^2} = \sqrt{5x^2 - 16x + 50}$$

$$\text{Let } g(x) = 5x^2 - 16x + 50 \Rightarrow g'(x) = 10x - 16 \Rightarrow g'(x) = 0 \Rightarrow x = \frac{8}{5}$$

The point $\left(\frac{8}{5}, -\frac{4}{5}\right)$ is the point on the graph closest to $(1, 3)$.

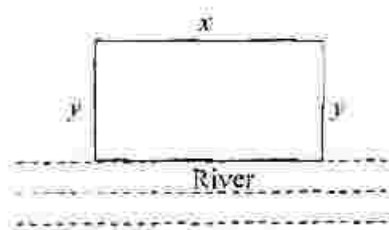
f. Exercise left for you.

22. **The shortest fence:** A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain $180,000 \text{ m}^2$ in order to provide enough grass for the herd. What dimensions would require the smallest amount of fencing if no fencing is needed along river.

Solution: i) Area = $xy \Rightarrow 180,000 \Rightarrow y = \frac{180,000}{x}$

ii) $2y + x \leftarrow$ quantity to be minimized (perimeter)

$$\Rightarrow A(x) = 2\left(\frac{180,000}{x}\right) + x = \frac{360,000}{x} + x$$



$$\Rightarrow A'(x) = 1 - \frac{360,000}{x^2} \Rightarrow A'(x) = 0$$

when $x = 600$

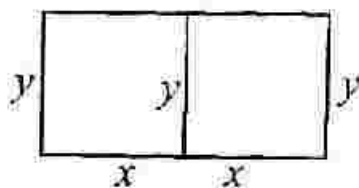
$$\text{Then } y = \frac{180,000}{x} = \frac{180,000}{600} = 300 \text{ m}$$

$\therefore$ The smallest amount of fencing when $y = 300 \text{ m}$ and $x = 600 \text{ m}$.

23. **Maximizing area:** A rancher (farmer) has 200 m of fencing with which to enclose two adjacent rectangular corrals (field). What dimensions should be used so that the enclosed area will be a maximum.

Solution: Perimeter = $4x + 3y = 200 \Rightarrow y = \frac{200}{3} - \frac{4}{3}x$

Area = $2xy \leftarrow$ quantity to be maximize



$$\Rightarrow A(x) = (2x)\left(\frac{200}{3} - \frac{4}{3}x\right) = \frac{400x}{3} - \frac{8x^2}{3}$$

$$\Rightarrow A'(x) = \frac{400}{3} - \frac{16x}{3} \Rightarrow A'(x) = 0 \quad \text{when}$$

$$x = \frac{400}{16} = 25, \text{ then } y = \frac{200}{3} - \frac{4}{3}x = \frac{200}{3} - \frac{4}{3}(25) = \frac{100}{3}$$

$\therefore x = 25 \text{ m}$ and $y = \frac{100}{3} \text{ m}$ yield maximum area.

24. **The maximum distance:** The mass connected to a spring moves along the x -axis so that its x -coordinate at time t is given by

$x(t) = \sin 2t + \sqrt{3} \cos 2t$. What is the maximum distance of the mass from the origin. (Gulick).

Solution: $x(t) = \sin 2t + \sqrt{3} \cos 2t$

$$\Rightarrow x'(t) = 2 \cos(2t) + -2\sqrt{3} \sin(2t) \Rightarrow x'(t) = 0 \text{ if } \cos 2t$$

$$= \sqrt{3} \sin 2t \Rightarrow \tan(2t) = \frac{1}{\sqrt{3}} \Rightarrow 2t = 30^\circ, 210^\circ$$

$$\Rightarrow t = 15^\circ \text{ and } t = 105^\circ \leftarrow \text{critical number.}$$

Now using 2nd derivative test

$$x''(t) = -4 \sin(2t) - 4\sqrt{3} \cos(2t), \text{ then } x''(150^\circ) = 0 \text{ and } x''(105^\circ)$$

$$= (-4) \left(-\frac{1}{2} \right) - 4\sqrt{3} \left(-\frac{\sqrt{3}}{2} \right) = 2 + 6 = 8 > 0.$$

Hence $t = 15^\circ$ gives maximum distance which is

$$x(15^\circ) = \sin(30^\circ) + \sqrt{3} \cos(30^\circ) = \frac{1}{2} + \frac{3}{2} = 2.$$

Thus the maximum distance of the mass from the origin is 2 unit.

Minimum Area: The combined perimeter of an equilateral triangle and a square is 10 m. Find the dimension of the triangle and the square that produce minimum total area.

Solution: Let x be the length of the side of triangle and y be the length side of square. Then $3x + 4y = 10 \Rightarrow y = \frac{10}{4} - \frac{3}{4}x$

$$\Rightarrow \text{Area of triangle} = \frac{1}{2}(x)x \sin 60^\circ = \frac{\sqrt{3}x^2}{4},$$

$$\text{Area of square} = y^2$$

$$\text{Area sum} = \frac{\sqrt{3}x^2}{4} + y^2 \Rightarrow A(x) = \frac{\sqrt{3}}{4}x^2 + \left(\frac{10}{4} - \frac{3}{4}x \right)^2$$

$$\Rightarrow A'(x) = \frac{\sqrt{3}}{2}x + 2 \left(\frac{10}{4} - \frac{3}{4}x \right) \left(-\frac{3}{4} \right)$$

$$\Rightarrow A'(x) = 0 \Rightarrow x = \frac{30}{4\sqrt{3} + 0} \text{ m}$$

$$\therefore x = \frac{10}{4\sqrt{3}+9} \text{ m: yield the minimum area}$$

26. A twelve meter wire is to be used to form circle and square. How much wire should be used for the circle and how much wire for the square if the total area enclosed is:
- a. minimum b. maximum

Solution:

$$\begin{array}{c} 2\pi r \quad 4x \\ \hline 12 \text{ m} \end{array}$$



The total area is given by

$$A = (\text{Area of square}) - (\text{Area of circle})$$

Let x be the side length and r is radius

$$\Rightarrow A = x^2 + \pi r^2 \dots (1).$$

Total length of wire is 12m (perimeter) given by: $P = (\text{perimeter of square}) + (\text{circumference of circle})$

$$\Rightarrow 12 = 4x + 2\pi r \Rightarrow 2\pi r = 12 - 4x \Rightarrow r = \frac{6-2x}{\pi}$$

$$\Rightarrow A(x) = x^2 + \pi \left[\frac{6-2x}{\pi} \right]^2 \text{ substituting } \frac{6-2x}{\pi}$$

$$\Rightarrow A'(x) = 2x + \frac{2\pi}{\pi^2} [6-2x] (-2) = 2x + \frac{8x-24}{\pi}, \quad (0 \leq x \leq 3)$$

$$\Rightarrow A'(x) = 0$$

$$\frac{2x\pi + 8x - 24}{\pi} = 0 \Rightarrow x = \frac{24}{2\pi + 8} = \frac{12}{\pi + 4} \text{ and}$$

$$r = \frac{6-2x}{\pi} = \frac{6}{\pi + 4} \leftarrow \text{radius of circle. Hence we conclude the}$$

sum of the area of circle and square minimum when the side of square is twice radius of circle.

Note, the maximum area occurs when $x = 0$. That is all the wire is used for the circle.

27. You have been asked to design to hold 1 L oil can shaped like a right circular cylinder can. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution: Volume of cylinder (v) = $\pi r^2 h \Rightarrow v = \pi r^2 h = 1000 \text{ cm}^3$

Applications of Differential Calculus

$$11 = 1000 \text{ cm}^3 \Rightarrow h = \frac{1000 \text{ cm}^3}{\pi r^2}$$

Surface area of can: $A = 2\pi r^2 + 2\pi rh \leftarrow$ quantity that minimize.

$$A(r) = 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2} \right) = 2\pi r^2 + \frac{2000}{r} \Rightarrow A'(r) = 4\pi r - \frac{2000}{r^2}$$



Cylinder

$$\frac{2000}{r^2} \Rightarrow A'(r) = 0 \text{ when } r = \sqrt[3]{\frac{500}{\pi}}. \text{ Applying 2}^{\text{nd}}$$

derivative test: $A''(r) = 4\pi + \frac{4000}{r^3} > 0$. Hence

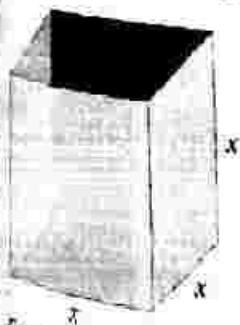
$r = \sqrt[3]{\frac{500}{\pi}}$ gives the minimum value. Thus

$$h = \frac{1000 \text{ cm}^3}{\pi r^2} = 2 \left(\sqrt[3]{\frac{500}{\pi}} \right) \leftarrow h = 2r$$

$\therefore$ To minimize the cost of the can, should be

$$h = 2 \left(\sqrt[3]{\frac{500}{\pi}} \right) \text{ and } r = \sqrt[3]{\frac{500}{\pi}}$$

23. Finding maximum value: A manufacture wants to design an open box having a square base and a surface area of 108 m^2 . What dimensions will produce a box with maximum volume?



Open box with square base

Solution: Since the box has a square base its volume is $v = x^2 h$...quantity to be optimized.

$$\Rightarrow s = (\text{Area of base}) + (\text{Area of 4 faces})$$

$$= x^2 + 4xh = 108 \Rightarrow h = \frac{108 - x^2}{4x} \text{ solving for } h \text{ in terms of } x.$$

$$\text{Now, } v = x^2 h \Rightarrow v(x) = x^2 \left(\frac{108 - x^2}{4x} \right) = 27x - \frac{x^3}{4}$$

$$\Rightarrow v'(x) = 27 - \frac{3x^2}{4} \Rightarrow v'(x) = 0.$$

Therefore, $27 - \frac{3x^3}{4} = 0 \Rightarrow 3x^3 = 108 \Rightarrow x = \pm 6$ is critical number. Accept $x = 6$, reject $x = -6$ (length is non-negative).

Using 2nd derivative $v''(x) = -\frac{6x}{4} \Rightarrow v''(6) = -\frac{36}{4} < 0$.

$\therefore v$ is maximum when $x = 6$, and $h = \frac{108 - x^3}{4x} = \frac{108 - 36}{24} = 3$

Examples from economics

Calculus makes important contribution to economic theory. First for maximizing profit and for minimize profit and for minimizing cost. Suppose that:

- $R(x)$ = the revenue from selling x item.
- $C(x)$ = the cost producing the x item.
- $P(x) = R(x) - C(x)$ = the profit from selling x item.

The marginal revenue, marginal and marginal profit at this production level (x - items) are:

- $\frac{dr}{dx}$ = marginal revenue
- $\frac{dc}{dx}$ = marginal cost
- $\frac{dp}{dx}$ = marginal profit

29. A company has examined its revenue structure given by

$R(x) = -\frac{1}{3}x^3 + 80x^2 + 72,000x$ where x is the number of units produced:

- What production level will yield a maximum revenue.
- What is the marginal revenue and when maximum.

Solution:

a. $R'(x) = -x^2 + 160x + 72,000$

$\Rightarrow R'(x) = -x^2 + 360x - 200x + 72,000$

$\Rightarrow R'(x) = -x(x - 360) - 200(x - 360) = (-x - 200)(x - 360)$

$\Rightarrow R''(x) = -2x + 160 \Rightarrow R''(360) = -720 + 160 < 0$

Therefore, $x = 360$ yield the maximum revenue.

Applications of Differential Calculus

Marginal revenue is $\frac{dR}{dx} = -x^2 + 160x + 72,000$ marginal

revenue is maximum when $-2x + 160 = 0 \Rightarrow x = 80$

A steel plant produces x tone of steel per week at total cost in birr

of $c(x) = \frac{1}{3}x^3 - 7x^2 + 111x + 8$. Find the marginal cost and its

minimum.

Solution: $C'(x) = x^2 - 14x + 111$...marginal cost

$C''(x) = 2x - 14 = 0 \Rightarrow x = 7$.

$\therefore$ The marginal cost is minimum when $x = 7$.

31. **Harmonic motion:** The height of an object attached to a spring is given by equation $x(t) = \frac{1}{3}\cos(12t) + \frac{1}{3}\sin(12t)$. Find the maximum displacement.

Solution: Here: $x'(t) = -4\sin(12t) + 4\cos(12t)$

$x'(t) = 0 \Rightarrow -4\sin(12t) + 4\cos(12t) = 0 \Rightarrow -\sin(12t) = -\cos(12t)$

$\Rightarrow \frac{\sin 12t}{\cos 12t} = \tan(12t) = 1 \Rightarrow 12t = \frac{\pi}{4} \Rightarrow t = \frac{\pi}{48}$

$\therefore$ The maximum displacement is

$$x\left(\frac{\pi}{48}\right) = \frac{1}{3}\cos\left(\frac{\pi}{4}\right) + \frac{1}{3}\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{3}$$

32. Find the area of the largest rectangle that can be inscribed in a circle of:

a. $r = 10$

b. $r = 9$

c. $r = 2$

Solution: Let the dimension of rectangle length be $2x$ and height be $2y$.

Consider the circle with center origin and radius r . Area of rectangle $A =$

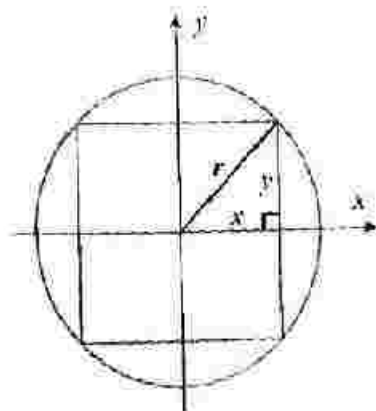
xy ...quantity optimized, $y = \sqrt{r^2 - x^2}$

$$\Rightarrow A(x) = x\sqrt{r^2 - x^2}$$

a. $r = 10$

$$\Rightarrow A(x) = x\sqrt{100 - x^2} = \sqrt{x^2} \sqrt{100 - x^2}$$

$$= \sqrt{100x^2 - x^4}$$



Applications of Differential Calculus

$$\text{Let } g(x) = 100x^2 - x^4 \Rightarrow g'(x) = 200x - 4x^3, g'(x) = 0$$

$$\Rightarrow 200x - 4x^3 = 4x(50 - x^2) = 0, x = 0 \text{ and } x = 5\sqrt{2} \quad \text{Applying 2nd derivative: } g''(x) = 200 - 12x^2$$

$$\therefore \text{when } x = 5\sqrt{2} \leftarrow \text{maximum.}$$

$$\therefore \text{The dimension of rectangle, length} = 2x = 10\sqrt{2} \text{ m and height } y = 2\sqrt{100 - 50} = 10\sqrt{2} \text{ m.}$$

$$\therefore \text{Area of rectangle} = 4xy = 800 \text{ m}^2 \leftarrow \text{maximum area.}$$

$$\text{b. } r = 9 \Rightarrow A(x) = \sqrt{81x^2 - x^4} \Rightarrow g(x) = 81x^2 - x^4$$

$$\Rightarrow g'(x) = 162x - 4x^3 \Rightarrow g'(x) = 0 \text{ when } 162x - 4x^3 = 0$$

$$\Rightarrow 2x(81 - 2x^2) \Rightarrow x = 0 \text{ and } x = \frac{9}{\sqrt{2}}, y = \frac{9}{2}$$

$$\therefore \text{length } 2x = \frac{18}{\sqrt{2}} \text{ and height } 2y = \frac{18}{\sqrt{2}}$$

$$\therefore \text{Area of rectangle: } 4xy = 4\left(\frac{18}{\sqrt{2}}\right)\left(\frac{18}{\sqrt{2}}\right) = 648 \text{ m}^2 \text{ is maximum area.}$$

c. Exercise left for you.

33. A window is constructed with top a semi circle and bottom is rectangle with total of perimeter of framing material is 16m. Find the dimension of the window to maximize the area.

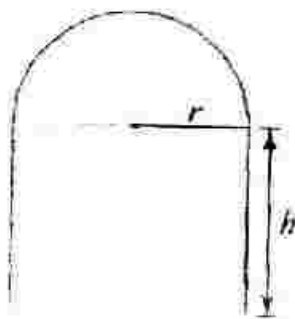
$$\text{Solution: Perimeter} = 2r + 2h + \pi r = 16 \text{ m} \Rightarrow h = 8 - r - \frac{1}{2}\pi r$$

$$\text{Area: } A = \frac{1}{2}\pi r^2 + 2rh \Rightarrow A(r) = \frac{1}{2}\pi r^2 + 2r\left(8 - r - \frac{1}{2}\pi r\right)$$

$$\Rightarrow A(r) = \frac{1}{2}\pi r^2 + 16r - 2r^2 - \pi r^2 = 16r - 2r^2 - \frac{1}{2}\pi r^2$$

$$\Rightarrow A'(r) = 16 - 4r - \pi r \Rightarrow A'(r) = 0 \Rightarrow 4r + \pi r = 16$$

$$\Rightarrow r = \frac{16}{4 + \pi} \text{ critical number and}$$



Applications of Differential Calculus

$$h = 8 - r - \frac{1}{2}\pi r = 8 - r \left(1 + \frac{1}{2}\pi\right) = 8 - \frac{16}{4 + \pi} \left(\frac{2 + \pi}{2}\right) = \frac{16}{4 + \pi}$$

An isosceles triangle has a perimeter of 16m. Find the length of the sides of a triangle that yield the maximum area.

Solution: Let the side length of x , x and y . Then,

Perimeter: $2x + y = 16$ and Area $= \frac{1}{2}hy$



$$h^2 + \frac{y^2}{4} = x^2$$

$$\Rightarrow h^2 = x^2 - \frac{y^2}{4} = \frac{4x^2 - y^2}{4} = \frac{(2x - y)(2x + y)}{4}$$

$$\Rightarrow h^2 = \frac{(16)(2x - y)}{4} \text{ put } 16 \text{ in } 2x + y$$

$$\Rightarrow h = 2\sqrt{2x - y} = 2\sqrt{(16 - y) - y} \text{ Put } 2x = 16 - y$$

$$\text{Area} = \frac{1}{2}hy \Rightarrow A(y) = \sqrt{16 - 2y} \cdot y = \sqrt{16y^2 - 2y^3} \dots \sqrt{y^2} = y$$

To optimize: let $g(y) = 16y^2 - 2y^3 \Rightarrow g'(y) = 32y - 6y^2$

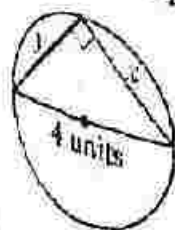
$$\Rightarrow g'(y) = 0 \Rightarrow 2y(16 - 3y) = 0 \Rightarrow y = 0 \text{ and } y = \frac{16}{3} \text{ critical number}$$

$$\Rightarrow x = \frac{16}{3} \text{ and } h = \frac{8\sqrt{3}}{3}$$

$\therefore$ The length of triangle should be $\frac{16}{3}, \frac{16}{3}$ and $\frac{16}{3}$ that yield maximum area which is $\frac{64}{9}\sqrt{3}\text{m}^2$.

35. What is the maximum area of a triangle which is inscribed in a circle of radius 2 units and one of its passing through the center of the circle (UEF)

A. 2 sq. unit B. 4π sq. unit C. 8 sq. unit D. 4 sq. unit



Solution: We have: $x^2 + y^2 = 16$ since one of side lies on center, $y = \sqrt{16 - x^2}$

Area of triangle (A) = $\frac{1}{2}xy$... to be optimized.

$$A(x) = \frac{1}{2}x\sqrt{16-x^2} = \frac{1}{2}\sqrt{16x^2-x^4}$$

Note: $x = \sqrt{x^2}, x \geq 0$

To optimize: let $g(x) = 16x^2 - x^4$

$$\Rightarrow g'(x) = 32x - 4x^3$$

$$g'(x) = 0 \Rightarrow 4x(8 - x^2) = 0 \Rightarrow x = 0 \text{ and } x = \sqrt{8}, y = \sqrt{8}$$

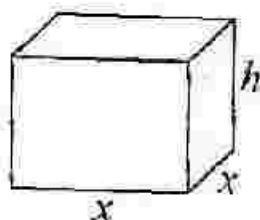
$$\therefore \text{Area (A)} = \frac{1}{2}(\sqrt{8})(\sqrt{8}) = 4 \text{ sq. unit.}$$

36. An open top square base rectangular box container is to have a volume of 32cm^3 . What should be the height (h) of the box to obtain the minimum possible surface area (UEE)

Answer: D

A. $h = 2 \text{ cm}$ B. $h = 8 \text{ cm}$ C. $h = 4 \text{ cm}$ D. $h = 16 \text{ cm}$

Solution: volume (v) = (base area) (height) = $x^2h = 32$, surface area = $x^2 + 4xh$ ← optimizing



$$A(x) = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}$$

$$\Rightarrow A'(x) = 2x - \frac{128}{x^2}$$

$$\Rightarrow A'(x) = 0 \Rightarrow 2x^3 = 128 \Rightarrow x^3 = 64 \Rightarrow x = 4.$$

$$\therefore \text{Therefore, } h = \frac{32}{x^2} = \frac{32}{16} = 2 \text{ cm}$$

∴ Answer: A

37. The total cost for ordering and storing x unit is given by

$C(x) = 3x + \frac{30,000}{x}$ birr. How many units must be ordered to produce a minimum cost.

A. 50 B. 100 C. 300 D. 1000

$$\text{Solution: } C'(x) = 3 - \frac{30,000}{x^2} \Rightarrow C'(x) = 0 \Rightarrow 3x^2 = 30,000$$

$$\Rightarrow x^2 = 10,000 \Rightarrow x = 100.$$

Applications of Differential Calculus

Answer: B

A closed cylindrical can is to hold 1 ℓ (1000 cm³) of liquid. How should we choose the height and radius to minimize the amount of material needed to manufacture the can.

Solution: Volume of cylinder (v) = $\pi r^2 h = 1000 \text{ cm}^3 \Rightarrow h = \frac{1000}{\pi r^2}$.

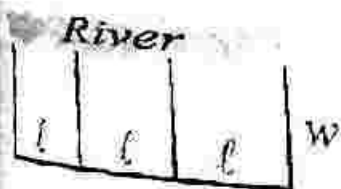
Surface: $2\pi r^2 + 2\pi r h$...quantity to optimize

$$\Rightarrow A(r) = 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2} \right) = 2\pi r^2 + \frac{2000}{r}$$

$$\Rightarrow A'(r) = 4\pi r - \frac{2000}{r^2} \Rightarrow A'(r) = 4\pi r - \frac{2000}{r^2}$$

$$A'(r) = 0 \Rightarrow r^3 = \frac{1000}{2\pi} \Rightarrow r = \frac{10}{\sqrt[3]{2\pi}} \text{ cm and } h = \frac{1000}{\pi r^2} = \frac{20}{\sqrt[3]{2\pi}}$$

39. A farmer wants to make three identical rectangular enclosure (same width and length) along a straight river as in the diagram show below. If he has a fencing material of 1,200 meters and the sides along the river need no fence



what should be the dimension (length ℓ and width w) of each enclosure if the total area of the enclosure is to be maximized...

(UEE)

A. $\ell = 100 \text{ m}, w = 225$ C. $\ell = 200 \text{ m}, w = 200 \text{ m}$

B. $\ell = 300 \text{ m}, w = 75$ D. $\ell = 200 \text{ m}, w = 150 \text{ m}$

Solution: Perimeter: $P = 3\ell + 4w = 1200 \Rightarrow w = 300 - \frac{3}{4}\ell$.

$$\text{Area (A)} = (3\ell)(w) \Rightarrow A(\ell) = 3\ell \left(300 - \frac{3}{4}\ell \right) = 900\ell - \frac{9}{4}\ell^2$$

$$\Rightarrow A'(\ell) = 900 - \frac{9}{2}\ell$$

$$A'(\ell) = 0 \Rightarrow 900 - \frac{9}{2}\ell = 0 \Rightarrow \ell = 200 \text{ m}$$

$$w = 300 - \frac{3}{4}(200) = 300 - 150 = 150 \text{ m}$$

∴ Answer: D

40. What are the value of a and b if $f(x) = \frac{ax^2 + b}{x^3}$ has extreme value at $(3, 2)$.

A. $a = 6, b = 3$

C. $a = 9, b = 27$

B. $a = 9, b = -27$

D. $a = -6, b = 4$

Solution: We have $f(3) = 2 \Rightarrow \frac{9a + b}{27} = 2 \Rightarrow 9a + b = 54$ and

$$f'(x) = \frac{2ax(x^3) - 3x^2(ax^2 + b)}{x^6} = \frac{-ax^2 - 3b}{x^4}$$

$$f'(3) = 0 \Rightarrow -9a - 3b = 0 \Rightarrow b = -3a, \text{ so that, } \begin{cases} 9a + b = 54 \\ b = -3a \end{cases}$$

$$\Rightarrow 6a = 54 \Rightarrow a = 9, b = -27$$

∴ Answer: B

41. What are the value of a and b if $f(x) = ax^3 + bx^4$ has minimum value at $(1, -1)$.

A. $a = 3, b = 4$

C. $a = -4, b = 3$

B. $a = 4, b = -3$

D. $a = 3, b = -4$

Solution: Exercise left for you (Answer: C).

42. A rectangular page contained 24 square inches of print. The margins

at the top and bottom of the page are to be $1\frac{1}{2}$ inch and the margins

on the left and the right to be 1 inches. What dimensions of the page will produce the smallest area in inches.

A. 6 by 4

B. 8 by 6

C. 9 by 6

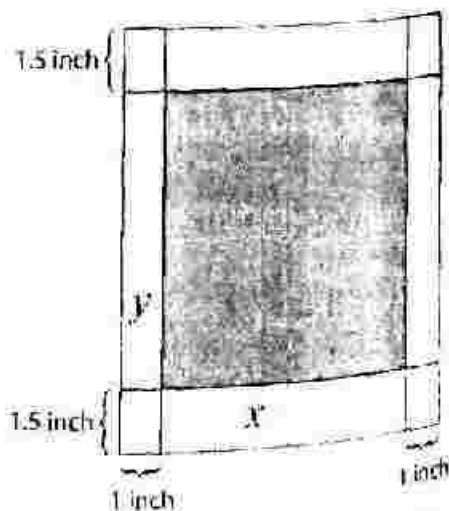
D. 10 by 8

Solution: Let A be area minimized:

$$A = (x + 2)(y + 3)$$

$$xy = 24 \dots \text{printed area inside margins}$$

$$\Rightarrow y = \frac{24}{x}$$



$$A(x) = (x-2) \left(\frac{24}{x} + 3 \right)$$

$$= 30 - 3x + \frac{48}{x}$$

$$A'(x) = 3 - \frac{48}{x^2} \Rightarrow A'(x) = 0 \Rightarrow 3x^2 - 48 = 0 \Rightarrow x^2 = 16 \Rightarrow x = 4 \text{ cm}$$

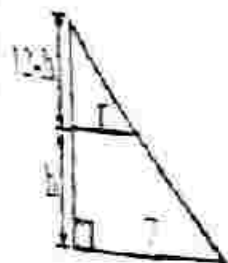
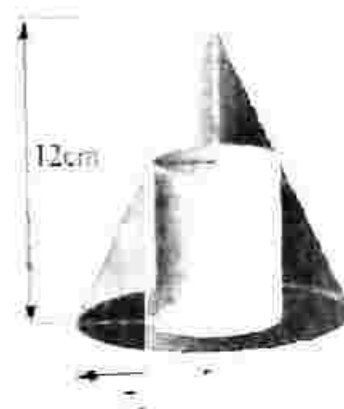
$$y = \frac{24}{x} = \frac{24}{4} = 6.$$

∴ Answer (

Find the radius and height of the cylinder of maximum volume that can be inscribed in a right circular cone of height 12 cm and radius of base 7 cm.

Solution: Let r = radius of cylinder, h = height of cylinder.

Then the volume of inscribed cylinder is given by



$V = \pi r^2 h$... to be optimized

$$\Rightarrow \frac{12-h}{r} = \frac{12}{7} \dots \text{using similarity property of triangle}$$

$$\Rightarrow h = 12 - \frac{12}{7}r \dots \text{solving for } h$$

$$\Rightarrow V(r) = \pi r^2 \left(12 - \frac{12}{7}r \right) = 12\pi r^2 - \frac{12}{7}\pi r^3, 0 \leq r \leq 7$$

$$\Rightarrow 12\pi r = 0 \text{ and } 2 - \frac{3}{7}r = 0 \Rightarrow r = \frac{14}{3} \text{ and } r = 0$$

$$\text{Thus, } h = 12 - \frac{12}{7} \left(\frac{14}{3} \right) = 12 - 8 = 4$$

∴ The inscribed cylinder has max. volume when $r = \frac{14}{3}$ cm

$h = 4$ cm
Related Rates

$$\Rightarrow A(x) = (x+2) \left(\frac{24}{x} + 3 \right)$$

$$= 30 + 3x + \frac{48}{x}$$

$$\Rightarrow A'(x) = 3 - \frac{48}{x^2} \Rightarrow A'(x) = 0 \Rightarrow 3x^2 - 48 = 0 \Rightarrow x^2 = 16 \Rightarrow x = 4 \text{ and}$$

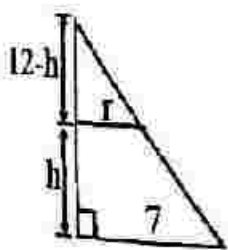
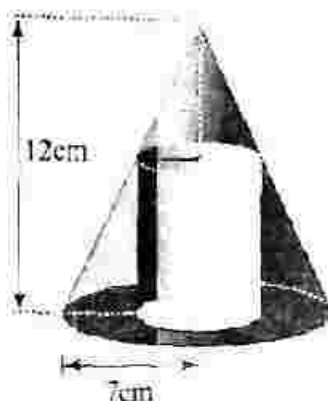
$$y = \frac{24}{x} = \frac{24}{4} = 6.$$

$\therefore$ Answer: C

43. Find the radius and height of the cylinder of maximum volume that can be inscribed in a right circular cone of height 12 cm and radius of base 7 cm.

Solution: Let r = radius of cylinder, h = height of cylinder.

Then the volume of inscribed cylinder is given by



$$v = \pi r^2 h \dots \text{to be optimized}$$

$$\Rightarrow \frac{12-h}{r} = \frac{12}{7} \dots \text{using similarity property of triangle}$$

$$\Rightarrow h = 12 - \frac{12}{7}r \dots \text{solving for } h$$

$$\Rightarrow v(r) = \pi r^2 \left(12 - \frac{12}{7}r \right) = 12\pi r^2 - \frac{12}{7}\pi r^3, 0 \leq r \leq 7$$

$$\Rightarrow 12\pi r = 0 \text{ and } 2 - \frac{3}{7}r = 0 \Rightarrow r = \frac{14}{3} \text{ and } r = 0.$$

$$\text{Thus, } h = 12 - \frac{12}{7} \left(\frac{14}{3} \right) = 12 - 8 = 4.$$

$\therefore$ The inscribed cylinder has max. volume when $r = \frac{14}{3}$ cm and

$h = 4$ cm.

Related Rates

Here we going to see the rates of change of two or more related variables that are changing with respect to time. For example: $y = x^3$, differentiation both side with respect to time.

$$\Rightarrow \frac{dy}{dt} = 3x^2 \frac{dx}{dt} \dots \text{both rate of change with respect to time is related.}$$

Two Rates that are related

Illustrative Example

44. The radius of a circle is increasing uniformly at the rate of 3 cm/sec. Find the rate of change of the area when:

- a. $r = 2$ cm b. $r = 6$ cm c. $r = 8$ cm

Solution: $A = \pi r^2$ differentiating with respect to time.

$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

a. $\frac{dA}{dt} = 2\pi(2 \text{ cm})3 \text{ cm/sec} = 12\pi \text{ cm}^2/\text{sec}$

b. When $r = 6 \Rightarrow \frac{dA}{dt} = 2\pi(6)3 \text{ cm/sec} = 36\pi \text{ cm}^2/\text{sec}$

c. When $r = 8 \text{ cm} \Rightarrow \frac{dA}{dt} = 2\pi(8 \text{ cm})(3 \text{ cm/sec}) = 48\pi \text{ cm}^2/\text{sec}$

45. The side of square sheet of metal is increasing at rate of 5 cm/min. Find the rate of change of the area when the is length x :

- a. $x = 4$ cm b. $x = 9$ cm c. $x = 6$ cm

Solution: Area of square $A = x^2 \Rightarrow \frac{dA}{dt} = 2x \frac{dx}{dt}$

a. $x = 4 \text{ cm} \Rightarrow \frac{dA}{dt} = 2(4 \text{ cm})5 \text{ cm/min} = 40 \text{ cm}^2/\text{min}$

b. $x = 9 \text{ cm} \Rightarrow \frac{dA}{dt} = 2(9)(5) \text{ cm}^2/\text{min} = 90 \text{ cm}^2/\text{min}$

c. $x = 6 \text{ cm} \Rightarrow \frac{dA}{dt} = 2(6)(5) \text{ cm}^2/\text{min} = 60 \text{ cm}^2/\text{min}$

46. The side of an equilateral triangle is increasing at the rate of 4 cm/sec. Find the rate of change of the area when.

- a. $x = \sqrt{3}$ cm b. $x = 8$ cm c. $x = 12$ cm

31

Solution: Let x be the side length of triangle:

$$\text{Area (A)} = \frac{1}{2}x^2 \sin 60^\circ \Rightarrow A \Rightarrow \frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2}x \frac{dx}{dt}, \text{ and}$$

given $\frac{dx}{dt} = 4 \text{ cm/sec}$

a. When $x = \sqrt{3} \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2}(\sqrt{3})(4) = 6 \text{ cm}^2/\text{sec}$

b. When $x = 8 \text{ cm} \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2}(8)(4) = 16 \text{ cm}^2/\text{sec}$

c. Exercise left for you.

47. A radius of spherical balloon is being inflated at the rate of 0.2 cm/sec . How fast is the volume increasing when the radius is

a. $r = 4 \text{ cm}$

b. $r = 3 \text{ cm}$

c. $r = 5 \text{ cm}$

Solution: $v = \frac{4}{3}\pi r^3$ the volume of sphere

$$\Rightarrow \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \text{ and } \frac{dr}{dt} = 0.2 \text{ cm/sec}$$

a. When $r = 4 \text{ cm} \Rightarrow \frac{dv}{dt} = 4\pi(16)(0.2) = 12.8\pi \text{ cm}^3/\text{sec}$

b. When $r = 3 \text{ cm} \Rightarrow \frac{dv}{dt} = 4\pi(3)^2(0.2) = 7.2\pi \text{ cm}^3/\text{sec}$

c. Exercise left for you.

48. A spherical balloon is inflated at the rate of $4 \text{ cm}^3/\text{min}$. How fast is the radius of the balloon increasing when the radius:

a. $r = 3 \text{ cm}$

b. $r = 6 \text{ cm}$

c. $r = 12 \text{ cm}$

Solution: Volume of sphere, $v = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \left(\frac{dv}{dt} \right) \text{ and } \frac{dv}{dt} = 4 \text{ cm}^3/\text{min}$$

a. When $r = 3 \text{ cm} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi(9)}(4) = \frac{1}{9\pi} \text{ cm/min}$

b. When $r = 6$ cm $\Rightarrow \frac{dr}{dt} = \frac{1}{(4\pi)(36)}(4) = \frac{1}{36\pi}$ cm/min

c. Exercise left for you.

49. The edge of cube is increasing at the rate of 0.5 cm/sec. At what rate is the surface area increasing when the edge of the cube is 10 cm.

Solution: Let x be the edge of cube. Then surface area: $A = 6x^2$

$$\Rightarrow \frac{dA}{dt} = 12x \frac{dx}{dt} = 12(10)0.5 = 60 \text{ cm}^2/\text{sec}$$

50. The edge of cube is 5 cm long and volume is increasing at the rate of $2 \text{ cm}^3/\text{min}$. How fast the surface of cube is increasing. (UFE)

a. $\frac{2}{5} \text{ cm}^2/\text{min}$ b. $\frac{8}{5} \text{ cm}^2/\text{min}$ c. $\frac{25}{5} \text{ cm}^2/\text{min}$ d. $8 \text{ cm}^2/\text{min}$

Solution: volume, $v = x^3$

$$\Rightarrow \frac{dv}{dt} = 3x^2 \frac{dx}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} \Rightarrow \frac{1}{3x^2} \left(\frac{dv}{dt} \right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{3(5)^2} 2 = \frac{2}{75} \text{ cm/min. And surface area, } A = 6x^2$$

$$\Rightarrow \frac{dA}{dt} = 12x \frac{dx}{dt} = 12(5) \left(\frac{2}{75} \right) = \frac{8}{5} \text{ cm}^2/\text{min}$$

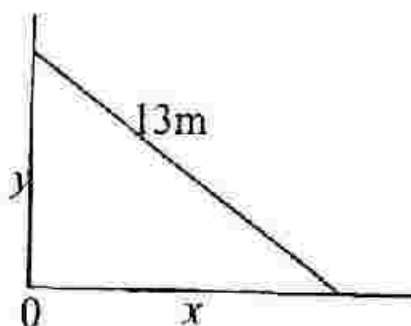
$\therefore$ Answer: B

51. A 13 m ladder is leaning against a wall of house! If the top of the ladder slides down the wall at a rate of 3 m/sec, how fast will the base (foot) be moving away from the wall when:

a. 5 m from the wall

b. $4\sqrt{3}$ m from the wall

Solution: At any time t , let $\overline{OA} = x$, $\overline{OB} = y$ so that $x^2 + y^2 = 13^2$



$$\Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \text{ and } \frac{dy}{dt} = 3 \text{ m/sec}$$

a. When $x = 5$ m $\Rightarrow 5^2 + y^2 = 13^2$

$$\Rightarrow y = 12 \text{ m}$$

$$\Rightarrow 2(5) \frac{dx}{dt} + 2(12)(3) = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{-72}{10} = -7.2 \text{ m/sec}$$

∴ The base (lower end) is moving 7.2 m/sec.

b. When $x = 4\sqrt{3} \text{ m} \Rightarrow (4\sqrt{3})^2 + y^2 = 169 \Rightarrow y^2 = 169 - 48 = 21$

$y = 11 \text{ m} \Rightarrow \frac{dx}{dt} = \frac{-y}{x} \frac{dy}{dt} = \frac{-11}{4\sqrt{3}} (3) = \frac{-11\sqrt{3}}{4} \text{ m/sec}$

∴ The base (lower end) is moving $\frac{11\sqrt{3}}{4} \text{ m/sec}$

2. A point moves along the curve $3y = x^2$ in such a way that the y -value is increasing at the rate of 4 unit per second. At what rate is x changing for each of the following values when:

- a. $x = \frac{1}{2}$ b. $x = 1$ c. $x = 8$

Solution: $3y = x^2 \Rightarrow 3 \frac{dy}{dt} = 2x \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{3}{2x} \frac{dy}{dt}$

a. When $x = \frac{1}{2} \Rightarrow \frac{dx}{dt} = \frac{3}{2} (2)(4) = 12 \text{ unit/sec} \leftarrow \frac{dy}{dt} = 4 \frac{\text{unit}}{\text{sec}}$

b. When $x = 1 \Rightarrow \frac{dx}{dt} = \frac{3}{2(1)} (4) = 6 \text{ unit/sec}$

c. Exercise left for you.

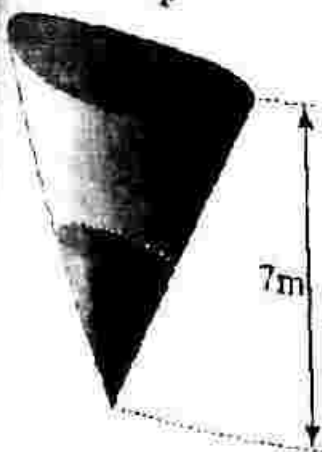
3. A water tank in the shape of an inverted circular cone with top radius of 3m and height 7m. If water is flowing into the tank at the rate of $3 \text{ m}^3/\text{min}$. Find the rate at which the water level is rising when the water is

a. 4m deep

b. 5m deep

Solution: Using similarity of triangle

$$\frac{r}{h} = \frac{3}{7} \Rightarrow r = \frac{3}{7} h \Rightarrow h = \frac{7}{3} r, \text{ Thus.}$$



a.

$$\Rightarrow \frac{dv}{dt} = \frac{9}{49} \pi h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{49}{9\pi h^2} \frac{dv}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{49}{9\pi(4)^2} \left(\frac{3\text{m}^3}{\text{min}} \right) = \frac{49}{48\pi} \text{m/min}$$

b. When $h = 5\text{m} \Rightarrow \frac{dh}{dt} = \frac{49}{9\pi(25)} (3\text{m}^3/\text{min}) = \frac{49}{75\pi} \text{m/min}$

54. A conical tank full of water is height 20 ft and radius 5 ft at the top. If the water is flowing out at the bottom at the rate of $2 \text{ ft}^3/\text{min}$. What is the rate at which the water level is falling when the water level is 16 ft above the bottom?

a. $\frac{8}{\pi} \text{ft/min}$ b. $\frac{1}{8\pi} \text{ft/min}$ c. $\frac{1}{18\pi} \text{ft/min}$ d. $\frac{18}{\pi} \text{ft/min}$

Solution: h = height water level at any time (t)

$$\Rightarrow \frac{5}{20} = \frac{r}{h} \Rightarrow \frac{r}{h} = \frac{1}{4} \Rightarrow r = \frac{1}{4}h \text{ and } \frac{dv}{dt} = 2 \text{ ft}^3/\text{min}$$

Volume of water level $v = \frac{1}{3} \pi r^2 h$ (since $r = \frac{1}{4}h$)

$$\Rightarrow v = \frac{1}{3} \pi \left(\frac{1}{4}h \right)^2 h = \frac{1}{48} \pi h^3 \Rightarrow \frac{dv}{dt} = \frac{1}{16} \pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{16}{\pi h^2} \left(\frac{dv}{dt} \right) = \frac{16}{\pi(16)^2} (2 \text{ ft}^3/\text{min}) = \frac{1}{8\pi} \text{ft/min}$$

$\therefore$ Answer: B

55. The radius of a right circular cylinder of height 10 cm increase at rate of 3 cm/min . What is the rate of change of the volume of cylinder when the radius is 8 cm?

A. $8\pi \text{cm}^3/\text{min}$

C. $480\pi \text{cm}^3/\text{min}$

B. $320\pi \text{cm}^3/\text{min}$

D. $640\pi \text{cm}^3/\text{min}$

Solution: We have $\frac{dv}{dt} = 3 \text{ cm/min}$, $v = \pi r^2 h$

$$\frac{dv}{dt} = 2\pi hr \frac{dr}{dt} = 20 \text{ cm} \pi (8 \text{ cm}) (3 \text{ cm/min}) = 480\pi \text{ cm}^3/\text{min}.$$

∴ Answer: C

Solved Problems

56. Which of the following is the set of critical number of

$$f(x) = |x^2 - 4|.$$

- A. $\{0\}$ B. $\{-2, 2\}$ C. $\{-2, 0, 2\}$ D. $\{0, 2\}$

$$\text{Solution: } f(x) = |x^2 - 4| = \begin{cases} x^2 - 4, & \text{if } x < -2 \text{ and } x > 2 \\ 0 & \text{if } x = -2 \text{ and } x = 2 \\ 4 - x^2 & \text{if } -2 < x < 2 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 2x & \text{if } x < -2 \text{ and } x > 2 \\ \text{Does not exist} & \text{if } x = -2 \text{ and } x = 2 \\ -2x & \text{if } -2 < x < 2 \end{cases} \Rightarrow f'(x) = 0 \text{ when } x = 0.$$

and $f'(x)$ does not exist when $x = -2$ and $x = 2$.

∴ $\{-2, 0, 2\}$ is set of critical number.

∴ Answer: C

57. Which one of the following is true about $f(x) = xe^{-x}$?

- A. f has a local min at $x = 1$
 B. f is strictly increasing on $[1, \infty)$
 C. the graph of f is concave down on $(-\infty, 2)$
 D. $(2, f(2))$ is not an inflection point

Solution: $f(x) = xe^{-x} \Rightarrow f'(x) = e^{-x}(1 - x)$. Then

i) $f'(x) > 0 \Rightarrow 1 - x > 0 \Rightarrow x < 1 \Rightarrow f$ is st. increasing on $(-\infty, 1)$.
 $x = 1$ is local max.

ii) $f''(x) = e^{-x}(x - 2) \Rightarrow f''(x) > 0 \Rightarrow x - 2 > 0 \Rightarrow x > 2$

$\Rightarrow$ The graph of f is concave up ward on $(2, \infty)$ and

$f''(x) < 0 \Rightarrow x < 2 \Rightarrow$ The graph concave down on $(-\infty, 2)$.

∴ Answer: C

58. The largest interval where $f(x) = \frac{x}{x^2 + 1}$ is increasing.

- A. $[0, \infty)$ B. $(-\infty, 1]$ C. $(1, \infty)$ D. $[-1, 1]$

Solution: $f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \Rightarrow \frac{1 - x^2}{(x^2 + 1)^2}$, then

$f'(x) > 0 \Rightarrow 1 - x^2 > 0 \Rightarrow -1 < x < 1 \Rightarrow f$ increasing on $(-1, 1)$.

∴ Answer: D

59. On which of the following intervals is the graph of $f(x) = \frac{x}{x^2 + 9}$ increasing... (UEE).

- A. $(-\infty, \infty)$ B. $[-3, 3]$ C. $(-\infty, -3]$ D. $[3, \infty)$

Solution: $f'(x) = \frac{x^2 + 9 - 2x^2}{(x^2 + 9)^2} = \frac{9 - x^2}{(x^2 + 9)^2}$, then

$$f'(x) < 0 \Rightarrow 9 - x^2 < 0 \Rightarrow -3 < x < 3 \Rightarrow f \text{ is increasing on } [-3, 3].$$

∴ Answer: B

60. Find the maximum possible area of rectangle with diagonal,

- a. $d = 8$ b. $d = 6$

Solution: $x^2 + y^2 = 8^2 \Rightarrow y^2 = 64 - x^2$

$$\text{Area } A = xy \Rightarrow A = x\sqrt{64 - x^2} \Rightarrow A(x) = \sqrt{64x^2 - x^4}$$

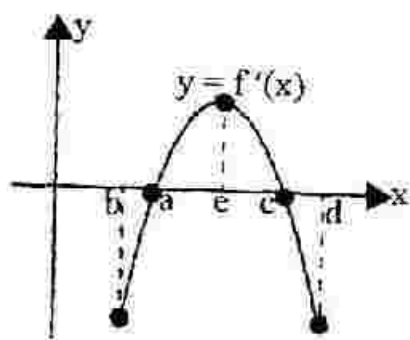
$$\text{Let } g(x) = 64x^2 - x^4 \Rightarrow g'(x) = 128x - 4x^3$$

$$g'(x) = 0 \text{ when } x = 0 \text{ and } x = \sqrt{32} \Rightarrow y = \sqrt{32}$$

∴ Area, $A = xy = \sqrt{32}\sqrt{32} = 32$ is maximum area.

b. Exercise left for you.

61. Suppose $f(x)$ is differentiable at all $x \in (-\infty, \infty)$. Then from the graph of $y = f'(x)$ given below. Find:



i) The interval on which the function f is increasing and decreasing.

ii) The concave structure and inflection point.

iii) Local extreme value of f

Solution: Recall, 1st derivative test.

i) Since $f'(x) > 0$, for all x in (a, e) for which f is increasing on $[a, e]$.

Also $f'(x) < 0$ for all x in (b, a) and (c, d) so that f is strictly decreasing on interval $[b, a]$ and $[c, d]$.

ii) Because the graph of f' is increasing on interval (b, e) so that the graph of f is *concave up* ward.

$\Rightarrow f''(x) > 0$ for all x in (b, e) . Similarly, the graph of f' is decreasing on interval (e, d) so that the graph of f is *concave down* ward on (c, d) .

$\Rightarrow f''(x) < 0$ for all x in (e, d) .

Inflection point $(e, f(e))$ since $f''(e) = 0$.

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- ii) Since $f'(a) = 0$ and $f''(a) > 0 \Rightarrow (a, f(a))$ is local min. and $f'(c) = 0$ and $f''(c) < 0 \Rightarrow (c, f(c))$ is local maximum.

62. Water is flowing into vertical cylindrical tank of radius 2m at the rate of $3\text{m}^3/\text{min}$. How fast is the water level rising.

- A. $\frac{2}{3\pi}\text{m/min}$ B. $\frac{3}{2\pi}\text{m/min}$ C. $\frac{3}{4\pi}\text{m/min}$ D. $\frac{4}{3\pi}\text{m/min}$

Solution: $v = \pi r^2 h \Rightarrow v = 4\pi h$, since $r = 2 \Rightarrow \frac{dv}{dt} = 4\pi \frac{dh}{dt}$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{4\pi} \left(\frac{dv}{dt} \right) = \frac{3}{4\pi} \text{m/min since } \frac{dv}{dt} = 3\text{m}^3/\text{min}$$

63. The minimum area of triangle formed by the x-axis and the y-axis and a line through the point (2, 1) is:

- A. 4 unit sq. B. 6 unit sq. C. 8 units sq. D. 12 units sq.

Solution: Let $(x, 0)$, $(0, y)$ and $(2, 1)$ point on the line. So that:

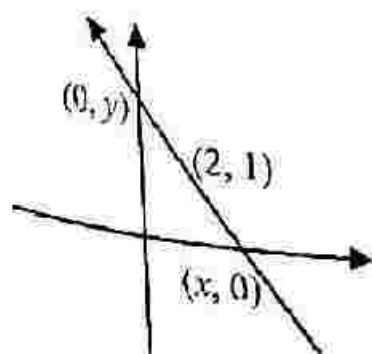
$$\frac{0-1}{x-2} = \frac{y-1}{0-2} \dots \text{from defn of slope of line}$$

$$\Rightarrow (x-2)(y-1) = 2 \Rightarrow xy - x - 2y = 0 \Rightarrow xy - x = 2y$$

$$\Rightarrow x(y-1) = 2y \Rightarrow x(y-1) = 2y \Rightarrow x = \frac{2y}{y-1}$$

Then, we optimize area

$$A = \frac{1}{2}xy \Rightarrow A(y) = \frac{1}{2} \left(\frac{2y}{y-1} \right) y = \frac{y^2}{y-1} \Rightarrow A'(y) = \frac{y^2 - 2y}{(y-1)^2}$$



$$A'(y) = 0 \Rightarrow y^2 - 2y = 0 \Rightarrow y = 2 \text{ and } x = 4$$

$$\therefore \frac{1}{2}xy = \frac{1}{2}(4)(2) = 4 \text{ units sq.}$$

$\therefore$ Answer: A

1. Find: i) **Supplementary Exercise on Unit - 13**
critical number

ii) absolute maximum value

iii) absolute minimum value

iv) the range of function, on the given interval

- a. $f(x) = x^3 - 12x, [0, 4]$ e. $f(x) = x + 2 \cos x, [0, 2\pi]$
- b. $f(x) = 3x^3 - 2x, [-1, 1]$ f. $f(x) = 3x^4 - 24x^2, [-3, 3]$
- c. $f(x) = x^2 - 2x, [-1, 3]$ g. $f(x) = 17 - 15x + 9x^2 - x^3, [0, 6]$
- d. $f(x) = \sqrt{12 - x - x^2}, [-4, 3]$ h. $f(x) = 3 - |x - 2|, [1, 4]$
2. Find: i) the interval in which f is st. increasing
 ii) the interval in which f is st. decreasing
 iii) local extreme value
- a. $f(x) = 3x^3 - 36x + 8$ f. $f(x) = 2x^4 + 3x^2 - 12x + 8$
- b. $f(x) = \frac{x}{x^2 + 16}$ g. $f(x) = x^3 e^{-x}$
- c. $f(x) = x^3 - 3x^2 + 16x + 3$ h. $f(x) = \frac{3x + 5}{x - 7}$
3. Find the value of a , b and k for which the function will have extreme value at indicated point.
- a. $f(x) = ax^3 - bx^2, \left(1, \frac{1}{2}\right)$ c. $f(x) = \frac{x^2 - 2x + 3k}{x + 2}, \text{ at } x = 1$
- b. $f(x) = x^4 + kx^3 + 10, x = 3$ d. $f(x) = \frac{x}{x^2 + k}, x = 3$
4. Determine the interval on which the graph has:
 i) concave upward ii) concave downward
 iii) locate the inflection, in each of the following.
- a. $f(x) = x^5$ c. $f(x) = \frac{4}{x^2 + 1}$ e. $f(x) = x^4 e^{2x}$
- b. $f(x) = x^6$ d. $f(x) = 5x^3 - 3x^2$
5. Find two positive number whose:
 a. sum 12, and product is maximum
 b. sum 15 and product is maximum
 c. product is 24 and whose sum is minimum
 d. sum: $x + y = 2$ and xy^2 is minimum or maximum
6. Find the length and width of rectangle with
 a. perimeter 18 m that maximize the area
 b. perimeter 120 m that maximize the area
 c. are 32m^2 that minimize the perimeter
 d. area 81m^2 that minimize the perimeter

7. Find the point on the graph of the function closest to the given point.

a. $f(x) = x^2, (0, 2)$

c. $f(x) = x^{\frac{1}{2}}, (4, 0)$

b. $f(x) = 4 - x^2, (0, 1)$

8. A cylindrical can is to be made to hold 1.5L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.
9. The combined perimeter of a circle and a square is 24m. Find the dimension of the circle and square that product a minimum total area.
10. The radius r of circle is increasing at a rate of 2 cm/sec. Find the rate of change of the area when: a) $r = 1$ cm b. $r = 5$ cm
11. A snow ball is melting at the rate of $1 \text{ cm}^3/\text{min}$. If it is always a sphere,
- How fast is the radius changing when the snow ball is 4 cm.
 - How fast is the surface area changing of the snow ball is 4 cm.
12. Find the dimension of the largest rectangle that can be inscribed in a semi circle of radius $r = 6$ cm.
13. A ladder of length 1^{st} , leans against house wall. If the top of the ladder slides down ward at the rate of 3 ft/sec. Find the rate at which its lower end is moving when it is left from the wall.
14. A spherical balloon is inflated at the rate of $3 \text{ cm}^3/\text{min}$. How fast is the radius of the balloon increasing when the radius: a) $r = 2$ cm b) $r = 6$ cm
15. Water is poured into a conical paper cup at the rate of $2 \text{ m}^3/\text{min}$ If the cup is 6 m tall and the top of the cup has radius of 2m, how fast does the water level rise when the water is 4m.

Unit Fourteen

I

ntegral Calculus

There are two kinds of integration. These are indefinite integral and the definite integral. Integration as reverse process of differentiation.

- **Integration** is the process of finding the original function $f(x)$ from its derivative $f'(x)$ less commonly called **the anti-derivative** which is denoted by elongated 's' $\int$.

- $x^3 + 1 \xrightarrow[\text{anti-derivative}]{\text{derivative}} 3x^2$ denoted by $\int 3x^2 dx = x^3 + c$

- $x^3 + 4 \xrightarrow[\text{anti-derivative}]{\text{derivative}} 3x^2$ denoted by $\int 3x^2 dx = x^3 + c$

- $F(x) + c \xrightarrow[\text{anti-derivative}]{\text{derivative}} f(x)$ denoted by

$$\int f(x) dx = F(x) + c$$

Thus, x^3 , $x^3 + 1$, ... and $x^3 + c$ are all the set of anti-derivative or integration of $3x^2$ write as $\int 3x^2 dx = x^3 + c$, the anti-derivative of f

(x) is not unique. The expression $\int f(x) dx$ is called an indefinite integral, where

i) $\int f(x) dx$ is the integral of $f(x)$ w.r.t x

ii) $f(x)$ is called integrand

iii) dx is called the variable of integration

iv) If $F'(x) = f(x)$ then $\int f(x) dx = F(x) + c$

The process of anti-differentiation is commonly called **integration**.

Basic Integration Rule

A. Integration of constant

$$\int k dx = kx + c \Rightarrow \frac{d}{dx}(kx + c) = k \text{ and } \int 0 dx = k \text{ Example:}$$

a. $\int 4 dx = 4x + c$

b. $\int c dx = cx + c$

$$c. \int \pi dx = \pi x + c$$

Integration of power function

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \Rightarrow \frac{d}{dx} \left(\frac{x^{n+1}}{n+1} + c \right) = x^n.$$

Example:

$$a) \int x^4 dx = \frac{x^5}{5} + c$$

$$c) \int x^\pi dx = \frac{x^{\pi+1}}{\pi+1} + c$$

$$b) \int x^{\frac{1}{2}} dx = \frac{2}{3} x^{\frac{3}{2}} + c$$

c. Integration of $\left(\frac{1}{x}\right)$

$$\int \frac{dx}{x} = \ln |x| + c, \text{ and } \int \frac{k dx}{x} = k \ln |x| + c.$$

Example: a) $\int \frac{dx}{2x} = \frac{1}{2} \ln |x| + c$ b) $\int \frac{3}{x} dx = 3 \ln |x| + c$

D. Integral of exponential function

$$i) \int e^x dx = e^x + c$$

$$ii) \int e^{mx} dx = \frac{e^{mx}}{m} + c$$

$$iii) \int a^x dx = \frac{a^x}{\ln a} + c$$

$$iv) \int a^{mx} dx = \frac{a^{mx}}{m \ln a} + c$$

Integral of trigonometric function:

$$i) \int \sin x dx = -\cos x + c$$

$$vi) \int \sec x \tan x dx = \sec x + c$$

$$ii) \int \cos x dx = \sin x + c$$

$$vii) \int \csc x \cot x dx = -\csc x + c$$

$$iii) \int \sec^2 x dx = \tan x + c$$

$$iv) \int \csc^2 x dx = \cot x + c$$

$$viii) \int \sin(kx) dx = -\frac{1}{k} \cos(kx) + c$$

$$x) \int \cos(kx) dx = \frac{1}{k} \sin(kx) + c$$

Illustrative Example

2. Find: a) $\int \sin 4x dx$ b) $\int \sin(5x-2) dx$
 c) $\int \sin(3-5x) dx$ d) $\int \cos(6x+1) dx$
 e) $\int \sec^2(3x+1) dx$ f) $\int \sec(2x) \tan 2x dx$
 g) $\int \csc^2(9x) dx$

Solution: a) $\int \sin 4x dx = -\frac{1}{4} \cos 4x + c$

b) $\int \sin(5x-2) dx = -\frac{1}{5} \cos(5x-2) + c$

c) $\int \sin(3-5x) dx = -\frac{1}{5} \cos(3-5x) + c$

d) $\int \cos(6x+1) dx = \frac{1}{6} \sin(6x+1) + c$

e) $\int \sec^2(3x+1) dx = \frac{1}{3} \tan(3x+1) + c$

f) $\int \sec(2x) \tan(2x) dx = \frac{\sec(2x)}{2} + c$

g) $\int \csc^2(9x) dx = -\frac{\cot 9x}{9} + c$

Properties of the indefinite integrals

i) $\int f'(x) dx = f(x) + c$ ii) $\frac{d}{dx} \int f(x) dx = f(x)$

iii) $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx \leftarrow \text{integrating term by term}$

Illustrative Example

3. Find: a) $\int \left(x^{-\frac{3}{4}} + x \right) dx$ b) $\int \sqrt[4]{x^5} dx$

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$$c) \int \left(x\sqrt{x} - 4\sqrt{x} + \frac{5}{\sqrt{x}} \right) dx$$

$$d) \int \frac{(x-2)}{\sqrt{x^3}} dx$$

$$e) \int \frac{5-e^x}{e^{2x}} dx$$

$$f) \int \left(\frac{x^3+3x}{x^2} \right) dx$$

$$g) \int \frac{(e^{2x} + 2e^x + 1)}{e^x}$$

$$h) \int e^x(1-e^x)^2 dx$$

Solution: Using the property it is the same as:

$$a) \int x^{\frac{-3}{4}} dx + \int x dx = \frac{x^{\frac{-3}{4}+1}}{\left(-\frac{3}{4}+1\right)} + \frac{x^2}{2} + 4x^{\frac{1}{4}} + \frac{x^2}{2} + c$$

$$b) \int x^{\frac{5}{4}} dx = \frac{x^{\frac{5}{4}+1}}{\frac{5}{4}+1} + c = \frac{4}{9} x^{\frac{9}{4}} + c = \frac{4}{9} \sqrt[4]{x^9} + c$$

$$c) \int x^{\frac{3}{2}} dx - 4 \int x^{\frac{1}{2}} dx + 5 \int x^{-\frac{1}{2}} dx = \frac{2}{5} x^{\frac{5}{2}} - \frac{8}{3} x^{\frac{3}{2}} + 10x^{\frac{1}{2}} + c$$

$$d) \int x^{\frac{1}{2}} dx - \int 2x^{-\frac{3}{2}} dx = 2x^{\frac{1}{2}} - 4x^{-\frac{1}{2}} + c$$

$$e) \int 5e^{-2x} dx - \int e^{-x} dx = \frac{-5}{2} e^{-2x} + e^{-x} + c \leftarrow \text{integrating term by term}$$

$$f) \int x dx + \int \frac{3}{x} dx = \frac{x^2}{2} + 3 \ln |x| + c \leftarrow \text{integrating term by term}$$

$$g) \int e^x + \int 2 dx + \int e^{-x} dx = e^x - e^{-x} + 2x + c$$

$$h) \int e^x(1-2e^x+e^{2x}) dx = \int e^x dx - \int e^{2x} dx = e^x - \frac{e^{2x}}{2} + c$$

Finding a function from its slope function and a point (Initial value problem)

Illustrative Example

4. Find the curve (function) whose slope at the point (x, y) is $3x^2 + 4$ if the curve is passing through the point $(-1, 3)$.

Solution: We have slope function: $\frac{dy}{dx} = 3x^2 + 4$

$$\Rightarrow dy = (3x^2 + 4) dx \Rightarrow$$

Since $(-1, 3)$ on the curve, so that $3 = (-1)^3 + 4(-1) + c$

$$\Rightarrow c + 3 + 4 = 8 \Rightarrow y = x^3 + 4x + 8 \leftarrow \text{the original function}$$

5. Find $f(x)$ satisfying the given function:

a. $f'(x) = x + \sin x$ and $f(0) = 4$

b. $f'(x) = x^2 + \frac{1}{x}$ and $f(1) = -\frac{5}{3}$

Solution: $f(x) = \int f'(x) dx = \int (x + \sin x) dx = x^2 - \cos x + c$

$$\Rightarrow f(0) = 0 - \cos 0 + c = 4 \Rightarrow c = 5 \Rightarrow f(x) = x^2 - \cos x + 5$$

b) $f(x) = \int f'(x) dx = \int \left(x^2 + \frac{1}{x} \right) dx = \frac{x^3}{3} + \ln |x| + c$

$$\Rightarrow f(1) = \frac{1}{3} + 0 + c = -\frac{5}{3} \Rightarrow c = \frac{-5-1}{3} = \frac{-6}{3} = -2$$

$$\therefore f(x) = \frac{x^3}{3} + \ln |x| - 2$$

Techniques of integration

There are four. These are:

- Integration by substitution.
- Integration by partial fraction.
- Integration by parts.
- Integration by tabular method.

A. Integration by substitution

The method of substitution *originated* by examining the rule of derivative backward.

Remember that: $\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$

$$\Rightarrow \int F'(g(x))g'(x) dx = F(g(x)) + c$$

$$\text{Let } u = g(x) \Rightarrow du = g'(x) dx$$

$\int F(u) du = F(u) + c \leftarrow$ It is called u -substitution.

Illustrative Example

Find:

a. $\int 6\sqrt[4]{4-3x} dx$ b. $\int \frac{3}{\sqrt{4-5x}} dx$ c. $\int (2x-3)^6 dx$

d. $\int \frac{3x^2+4x}{x^3+2x^2+7} dx$ e. $\int \frac{x dx}{\sqrt{3x^2+1}}$ f. $\int \frac{x+2}{\sqrt{x-2}} dx$

g. $\int 3x\sqrt{6x^3+11} dx$ h. $\int x^2\sqrt{x-1} dx$ i. $\int x^5\sqrt{x^2-1} dx$

j. $\int \sqrt{2x-3} dx$ k. $\int \sqrt[3]{1-2x} dx$ l. $\int (\pi-5x)^e dx$

Solution: a) Let $u = 4 - 3x \Rightarrow du = -3dx \Rightarrow dx = \frac{-du}{3}$

$$\therefore \int 6\sqrt[4]{4-3x} dx = -\int 6\sqrt[4]{u} \frac{du}{3} = -2 \int u^{\frac{1}{4}} du$$

$$= \frac{-8}{5} [u]^{\frac{5}{4}} + c = \frac{-8}{5} \left(\sqrt[4]{4-3x} \right)^5 + c$$

b) Let $u = 4 - 5x \Rightarrow du = -5dx \Rightarrow dx = \frac{-du}{5}$

$$\therefore \int \frac{3}{\sqrt{4-5x}} dx = \frac{-1}{5} \int \frac{3}{\sqrt{u}} du$$

$$= \frac{-3}{5} \int u^{-\frac{1}{2}} du = \frac{-18}{25} [u]^{\frac{5}{2}} + c = \frac{-18}{25} \left(\sqrt{4-5x} \right)^5 + c$$

c) Let $u = 2x - 3 \Rightarrow du = 2dx \Rightarrow dx = \frac{du}{2}$

$$\therefore \int (2x-3)^6 dx = \int u^6 \frac{du}{2} = \frac{1}{2} \left[\frac{u^7}{7} \right] = \frac{1}{14} (2x-3)^7 + c$$

d) Let $u = x^3 + 2x^2 + 7 \Rightarrow du = (3x^2 + 4x) dx$

$$\therefore \int \frac{(3x^2+4x)dx}{x^3+2x^2+7} = \int \frac{du}{u} = \ln|u| + c = \ln|x^3+2x^2+7| + c$$

e) Let $u = 3x^2 + 1 \Rightarrow du = 6x dx \Rightarrow x dx = \frac{du}{6}$

$$\begin{aligned}\therefore \int \frac{x dx}{\sqrt{3x^2 + 1}} &= \frac{1}{6} \int \frac{du}{\sqrt{u}} = \frac{1}{6} \int u^{-\frac{1}{2}} du = \frac{1}{3} u^{\frac{1}{2}} + c \\ &= \frac{1}{3} \sqrt{3x^2 + 1} + c\end{aligned}$$

f) Let $u = \sqrt{x-2} \Rightarrow u^2 = x-2 \Rightarrow x = u^2 + 2 \Rightarrow dx = 2u du$

$$\begin{aligned}\therefore \int \frac{x+2}{\sqrt{x-2}} dx &= 2 \int \frac{(u^2 + 2) + 2}{u} u du = 2 \int (u^2 + 4) du \\ &= 2 \left(\frac{u^3}{3} + 4u \right) + c = \frac{2\sqrt{x-2}^3}{3} + 8\sqrt{x-2} + c\end{aligned}$$

g) Let $u = 6x^3 + 11 \Rightarrow du = 18x^2 dx \Rightarrow x^2 dx = \frac{du}{18}$

$$\therefore \int 3x^2 \sqrt{6x^3 + 11} dx = \frac{1}{6} \int \sqrt{u} du = \frac{1}{9} u^{\frac{3}{2}} + c = \frac{1}{9} \left(\sqrt{6x^3 + 11} \right)^3 + c$$

h) Let $u = \sqrt{x-1} \Rightarrow u^2 = x-1 \Rightarrow x = u^2 + 1 \Rightarrow dx = 2u du$

$$\begin{aligned}\therefore \int x \sqrt{x-1} dx &= \int (u^2 + 1)u(2u du) \\ &= 2 \int (u^4 + u^2) du = \frac{2}{5} u^5 + \frac{2}{3} u^3 + c \\ &= \frac{2}{5} (\sqrt{x-1})^5 + \frac{2}{3} (\sqrt{x-1})^3 + c\end{aligned}$$

i) Let $u = \sqrt{x^2 + 1} \Rightarrow u^2 = x^2 + 1 \Rightarrow x^2 = u^2 - 1 \Rightarrow 2x dx = 2u du$

$$\begin{aligned}\therefore \int (x^2)^2 \sqrt{x^2 + 1} x dx &= \int (u^4 + 2u + 1)u^2 du = \frac{u^7}{7} + \frac{2u^4}{4} + \frac{u^3}{3} + c \\ &= \frac{\sqrt{x^2 + 1}^7}{7} + \frac{1}{2} \sqrt{x^2 + 1}^4 + \frac{\sqrt{x^2 + 1}^3}{3} + c\end{aligned}$$

$$\text{Let } u = 2x - 3 \Rightarrow du = 2dx \Rightarrow dx = \frac{du}{2}$$

$$\therefore \int \sqrt{2x-3} dx = \int \sqrt{u} \frac{du}{2} = \frac{1}{3} u^{\frac{3}{2}} + c = \frac{1}{3} \sqrt{2x-3} + c$$

Exercise left for you.

$$\text{Let } u = \pi - 5x \Rightarrow du = -5dx \Rightarrow dx = \frac{-du}{5}$$

$$\therefore \int (\pi - 5x)^e dx = \int u^e \frac{du}{5} = \frac{-u^{e+1}}{5e+5} = \frac{-(\pi - 5x)^{e+1}}{5e+5} + c$$

Find: a) $\int (x^2 + 2x + 1)^6 dx$ b) $\int (9x^2 - 12x + 4)^5$

c) $\int (25x^2 - 10x + 1) dx$

Solution: a) $x^2 + 2x + 1 = (x+1)^2 \Rightarrow \text{let } u = x+1 \Rightarrow du = dx$

$$\therefore \int (x^2 + 2x + 1)^6 dx = \int (x+1)^{12} dx$$

$$= \int u^{12} du = \frac{u^{13}}{13} + c = \frac{(x+1)^{13}}{13} + c$$

b) $9x^2 - 12x + 4 \Rightarrow (3x-2)^2 \Rightarrow u = 3x-2 \Rightarrow du = 3 dx$

$$\therefore \int (9x^2 - 12x + 4)^5 dx = \int (3x-2)^{10} dx = \int u^{10} \frac{du}{3}$$

$$= \frac{u^{11}}{33} + c = \frac{(3x-2)^{11}}{33} + c$$

c) $25x^2 - 10x + 1 = (5x-1)^2 \Rightarrow u = 5x-1 \Rightarrow du = 5dx$

$$\therefore \int (25x^2 - 10x + 1)^8 dx = \int (5x-1)^{16} dx$$

$$= \int u^{16} \frac{du}{5} = \frac{u^{17}}{85} + c = \frac{(5x-1)^{17}}{85} + c$$

Substitution involving exponential

Find:

a. $\int \frac{dx}{1+e^{-x}}$

b. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

c. $\int \frac{e^x}{x^2} dx$

$$d. \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$e. \int e^x \sqrt{e^x + 1} dx$$

$$f. \int 2^x (1 - 2^{x+1})^6 dx \quad g. \int 3^{x\sqrt{x}} \sqrt{x} dx$$

$$\text{Solution: } \int \frac{dx}{1 + e^{-x}} = \int \frac{e^x dx}{1 + e^x}, \text{ let } u = 1 + e^x \Rightarrow du = e^x dx$$

$$\therefore \int \frac{du}{u} = \ln |u| + c = \ln |1 + e^x| + c$$

$$b) \text{ Let } u = e^x + e^{-x} \Rightarrow du = (e^x - e^{-x}) dx$$

$$\Rightarrow \int \frac{(e^x - e^{-x})}{e^x + e^{-x}} dx = \int \frac{du}{u} = \ln |u| + c = \ln(e^x + e^{-x}) + c$$

$$c) \text{ Let } u = \frac{1}{x} \Rightarrow du = \frac{-1}{x^2} dx = \frac{dx}{x^2} = -du$$

$$\Rightarrow \int \frac{e^{\frac{1}{x}}}{x^2} dx = - \int e^u du = -e^u + c = -e^{\frac{1}{x}} + c$$

$$d) \text{ Let } u = \sqrt{x} \Rightarrow du = \frac{dx}{\sqrt{x}}$$

$$\therefore \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int e^u du = e^u + c = e^{\sqrt{x}} + c$$

$$e) \text{ Let } u = e^x + 1 \Rightarrow du = e^x dx$$

$$\therefore \int e^x \sqrt{e^x + 1} dx = \int \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} + c = \frac{2}{3} \sqrt{e^x + 1}^3 + c$$

$$f) \text{ Let } u = 1 - 2^{x+1} \Rightarrow du = -2^x (2 \ln 2) dx \Rightarrow \frac{-du}{2 \ln 2} = 2^x dx$$

$$\begin{aligned} \therefore \int (1 - 2^{x+1})^6 (2^x dx) &= - \int u^6 \frac{du}{2 \ln 2} = - \left(\frac{u^7}{7 \ln 4} \right) + c \\ &= \frac{(1 - 2^{x+1})^7}{7 \ln 4} + c \end{aligned}$$

$$g) \text{ Let } u = x\sqrt{x} = x^{\frac{3}{2}} \Rightarrow du = \frac{3}{2}x^{\frac{1}{2}}dx = \frac{3}{2}\sqrt{x}dx$$

$$\Rightarrow \int 3^{x\sqrt{x}} \sqrt{x} dx = \frac{2}{3} \int 3^u du = \frac{2(3^u)}{3 \ln 3} = \frac{2(3^{x\sqrt{x}})}{3 \ln 3} + c$$

Substitution involving natural logarithm

Illustrative Example

9. Find:

a. $\int \frac{\ln x}{x} dx$ b. $\int \frac{(\ln x)^3}{x} dx$ c. $\int \frac{(\ln x)}{x} dx$ d. $\int \frac{dx}{x(\ln x)^2}$

Solution: Here a) Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$\therefore \int \frac{\ln x}{x} dx = \int u du = \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$$

b) Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$\therefore \int \frac{(\ln x)^3}{x} dx = \int u^3 du = \frac{u^4}{4} + c = \frac{(\ln x)^4}{4} + c$$

c) Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$$\therefore \int \frac{(\ln x)^n}{x} dx = \int u^n du = \frac{u^{n+1}}{n+1} + c = \frac{(\ln x)^{n+1}}{n+1} + c$$

d) Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$$\therefore \int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = \int u^{-2} du = -\frac{1}{u} + c = -\frac{1}{\ln x} + c$$

Note: Substitution involving trigonometric

- $\int \tan x dx = -\ln |\cos x| + c$, use $u = \cos x$
- $\int \cot x dx = \ln |\sin x| + c$, use $u = \sin x$

- $\int \sin kx \, dx = -\frac{\cos kx}{k} + c$, use $u = kx$
- $\int \cos kx \, dx = \frac{\sin kx}{k} + c$, use $u = kx$

Illustrative Example

10. Find:

- a. $\int \sin(4-5x) \, dx$ b. $\int \sin x \, dx$ c. $\int e^x \sin(e^x) \, dx$
 d. $\int \sin^3 x \cos x \, dx$ e. $\int \cos^n x \sin x \, dx$ f. $\int \frac{1}{x^2} \cos\left(\frac{1}{x}\right) - 1 \, dx$
 g. $\int \frac{e^x \, dx}{\cos^2(e^x + 3)}$ h. $\int \cos^4 x \sin x \, dx$ i. $\int x^2 \sin(x^3) \, dx$

Solution: a) Let $u = 4 - 5x \Rightarrow du = -5dx \Rightarrow dx = -\frac{du}{5}$

$$\int \sin(4-5x) \, dx = -\int \sin u \frac{du}{5} = -\frac{\cos u}{5} + c = -\frac{\cos(4-5x)}{5} + c$$

b) Let $u = 1 + 3 \cos x \Rightarrow du = -3 \sin x \, dx \Rightarrow \sin x \, dx = -\frac{du}{3}$

$$\therefore \int \frac{\sin x}{1+3 \cos x} = -\int \frac{du}{3u} = -\frac{1}{3} \ln |u| + c = -\frac{1}{3} \ln |1 + \cos x| + c$$

c) Let $u = e^x \Rightarrow du = e^x \, dx \Rightarrow \int \sin u \, du = -\cos u + c = -\cos(e^x) + c$

d) Let $u = \sin x \Rightarrow du = \cos x \, dx \Rightarrow \int u^3 \, du = \frac{u^4}{4} + c = \frac{\cos^4 x}{4} + c$

e) Let $u = \cos x \Rightarrow du = -\sin x \, dx$

$$\therefore \int \cos^n x \sin x \, dx = \int u^n \, du = -\frac{u^{n+1}}{n+1} + c = -\frac{\cos^{n+1} x}{n+1} + c$$

f) Let $u = \frac{1}{x} - 1 \Rightarrow du = -\frac{1}{x^2} \, dx \Rightarrow \frac{dx}{x^2} = -\frac{du}{1}$

$$\therefore \int \frac{1}{x^2} \cos\left(\frac{1}{x} - 1\right) \, dx = -\int \cos u \, du = -\sin u + c = -\sin\left(\frac{1}{x} - 1\right) + c$$

$$\text{Let } u = e^x + 3 \Rightarrow du = e^x dx$$

$$\therefore \int \frac{e^x dx}{\cos^2(e^x + 3)} = \int \frac{du}{\cos^2 u} = \int \sec^2 u du = \tan u + c$$

$$= \tan(e^x + 3) + c$$

b) Exercise left for you. Use, $u = \cos x$

$$\text{c) Let } u = x^3 \Rightarrow du = 3x^2 dx \Rightarrow x^2 dx = \frac{du}{3}$$

The integrals of $\sin^2 x$ and $\cos^2 x$

$\sin^2 x$ and $\cos^2 x$ arise frequently in application of integrals.

$$\text{Recall: i) } \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\text{ii) } \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

1. Find:

a. $\int \sin^2 x dx$

b. $\int \cos^2 x dx$

c. $\int \sin^2 3x dx$

d. $\int \sin^3 x dx$

e. $\int \cos^3 x dx$

f. $\int \sin^4 x dx$

Solution: Use identity property of trigonometry.

a) $\int \sin^2 x dx = \int \frac{(1 - \cos 2x)}{2} dx = \frac{1}{2}x - \frac{\sin 2x}{4} + c$

b) $\int \cos^2 x dx = \int \frac{1 + \cos(2x)}{2} dx = \frac{1}{2}x + \frac{\sin 2x}{4} + c$

c) $\int \sin^2 3x = \int \frac{1 - \cos 6x}{2} dx = \frac{1}{2}x - \frac{\sin 6x}{12} + c$

d) $\int \sin^3 x dx = \int \sin^2 x \sin x dx = \int (1 - \cos^2 x) \sin x dx$

Let $u = \cos x \Rightarrow du = -\sin x dx$ and $\int \sin^3 x dx = \int (1 - \cos^2 x) \sin x dx$

$$\Rightarrow -\int (1 - u^2) du = -u + \frac{u^3}{3} + c = -\cos x + \frac{\cos^3 x}{3} + c$$

12. $\int \frac{dx}{x + \sqrt{x}}$ is equal to:

- A. $\frac{\ln(x+1)}{2} + c$ C. $\frac{\ln(1+\sqrt{x}) + c}{2}$
 B. $2\ln(\sqrt{x}+1) + c$ D. $\ln\sqrt{x+\sqrt{x}} + c$

Solution: Let $u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow dx = 2u du$

$$\begin{aligned} \therefore \int \frac{dx}{x+\sqrt{x}} &= \int \frac{2u du}{u^2+u} = 2 \int \frac{du}{u+1} \\ &= 2 \ln|u+1| + c = 2 \ln(\sqrt{x}+1) + c \quad \therefore \text{Answer: B} \end{aligned}$$

13. $\int \frac{\log_3^x}{x} dx$ is equal to:

- A. $(\ln 3)(\ln x)^2 + c$ B. $\frac{(\ln x)^2}{\ln 3} + c$ C. $\left(\frac{\ln x}{\ln 3}\right)^2 + c$ D. $(\ln 3) \ln x + c$

Solution: $\log_3^x = \frac{\log_e^x}{\log_e^3} = \frac{\ln x}{\ln 3}$ thus $\int \frac{\log_3^x}{x} dx = \int \frac{\ln x}{x \ln 3} dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$$\therefore \int \frac{u du}{\ln 3} = \frac{1}{\ln 3} u^2 + c = \frac{(\ln x)^2}{\ln 3} + c \quad \therefore \text{Answer: B}$$

14. $\int 2^x \sqrt{1+2^x} dx$ is equal to:

- A. $\frac{2}{3 \ln 2} (\sqrt{1+2^x})^3 + c$ C. $\frac{3}{2 \ln 2} (\sqrt{1+2^x})^3 + c$
 B. $\frac{2 \ln 2}{3} (1+2^x)^3$ D. None

Solution: $u = 1 + 2^x \Rightarrow du = 2^x \ln 2 dx \Rightarrow \frac{du}{\ln 2} = 2^x dx$

$$\Rightarrow \int \sqrt{u} \frac{du}{\ln 2} = \frac{1}{\ln 2} \left(\frac{2}{3}\right) u^{\frac{3}{2}} + c = \frac{2}{3 \ln 2} (\sqrt{1+2^x}) + c \therefore$$

Answer: A

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15. $\int \frac{2^{1-3x}}{3^{x+1}} dx$ is equal to:

A. $\frac{2(24)^x}{3 \ln 24} + c$

C.

$$\frac{-2(24)^x}{3 \ln 24} + c$$

B. $\frac{-2(24)^{-x}}{3 \ln 24} + c$

D.

$$\frac{2 \ln 24 (24)^{-x}}{3} + c$$

Solution: $\frac{2^{1-3x}}{3^{x+1}} = \frac{2^1}{2^{3x}} \cdot \frac{1}{3 \cdot 3^x} = \frac{2}{3} \cdot \frac{1}{(24)^x} = \frac{2}{3} \cdot (24)^{-x}$

$$\therefore \int \frac{2}{3} (24)^{-x} dx = \frac{-2(24)^{-x}}{3 \ln 24} + c$$

$\therefore$ Answer: B

16. $\int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x} dx$ is equal to:

A. $\frac{1}{3} \ln |x^2 + 2x + 3| + c$

B. $\frac{1}{3} \ln |x^3 + 3x^2 + 9x| + c$

C. $\ln |x+3| - \ln |x+1| + c$

Solution: Let $u = x^3 + 3x^2 + 9x \Rightarrow du = (3x^2 + 6x + 9) dx$

$$\therefore \int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x} = \int \frac{du}{3u} = \frac{1}{3} \ln |u| + c = \frac{1}{3} \ln |x^3 + 3x^2 + 9x| + c$$

$\therefore$ Answer: B

17. $\int \frac{dx}{x \log_{10} x}$ is equal to:

A. $\ln 10 \ln |\ln x| + c$

B. $\frac{\ln |\ln x|}{\ln 10} + c$

C. $\frac{\ln x}{\ln 10} + c$

D. $(\ln 10) \ln x + c$

Solution:

$$u = \log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\ln x}{\ln 10} \Rightarrow du = \frac{dx}{x \ln 10} = (\ln 10) du = \frac{dx}{x}$$

$$\therefore \int \frac{dx}{x \log_{10} x} = \int \frac{(\ln 10) du}{u} = (\ln 10) \ln |u| + c = (\ln 10) \ln |\ln x| + c$$

Answer: A

18. $\int \cos^3 x \, dx$ is equal to ...

- A. $\cos x - \frac{\cos^3 x}{3} + c$ C. $\sin x - \frac{\cos^2 x}{2} + c$
 B. $\sin x - \frac{\sin^3 x}{3} + c$ D. None

Solution: $\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$

let $u = \sin x \Rightarrow du = \cos x \, dx \Rightarrow$

$$\int (1 - u^2) du = u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c$$

B. Integration by partial fraction

This section examines a method for decomposing rational function into simpler rational function to apply the basic integration rule. This method is called **method of partial fraction**.

Linear factor rule:

a. **Integration of distinct linear fraction**

$$\frac{N(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d} \Leftrightarrow A(cx+d) + B(ax+b) = N(x)$$

Illustrative Example

19. Find:

a. $\int \frac{dx}{x(x+2)}$ b. $\int \frac{dx}{x^2-5x+6}$ c. $\int \frac{(4x+1)}{x^2-3x+2} dx$
 d. $\int \frac{x}{x^2+4x+3} dx$ e. $\int \frac{12}{x^2-9} dx$ f. $\int \frac{4x+5}{x^2+x-2} dx$

Solution: a) $\frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow A(x+2) + Bx = 1,$

multiplying both side by $x(x+2)$. **To find for A:** set $x = 0 \Rightarrow$

$2A = 1 \Rightarrow A = \frac{1}{2}$. **To find for B:** set $x = -2 \Rightarrow -2B = 1, B = -\frac{1}{2}$

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$$\begin{aligned}\therefore \int \frac{dx}{x(x+2)} &= \frac{1}{2} \int \frac{dx}{x} - \frac{1}{2} \int \frac{dx}{x+2} \\ &= \frac{1}{2} \ln|x| - \frac{1}{2} \ln|x+2| + c = \frac{1}{2} \ln \left| \frac{x}{x+2} \right| + c\end{aligned}$$

$$\begin{aligned}\text{b) } \frac{1}{x^2 - 5x + 6} &= \frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \\ \Leftrightarrow A(x-3) + B(x-2) &= 1\end{aligned}$$

To find for A: set $x = 2 \Rightarrow A(2-3) + 0 = 1 \Rightarrow A = -1$.

To find for B: set $x = 3 \Rightarrow 0 + B(3-2) = 1 \Rightarrow B = 1$

$$\begin{aligned}\therefore \int \frac{dx}{(x-2)(x-3)} &= \int \frac{-dx}{x-2} + \int \frac{dx}{x-3} \\ &= -\ln|x-2| + \ln|x-3| + c = \ln \left| \frac{x-3}{x-2} \right| + c\end{aligned}$$

$$\text{c) } \frac{4x+1}{x^2 - 3x + 2} = \frac{4x+1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$$\Leftrightarrow A(x-1) + B(x+2) = 4x+1.$$

To find for A: let $x = 2 \Rightarrow A(2-1) + 0 = 4(2) + 1 \Rightarrow A = 9$.

To find for B: set $x = 1 \Rightarrow 0 + B = 4 + 1 \Rightarrow B = 5$

$$\therefore \int \frac{4x+1}{(x-2)(x-1)} = 9 \int \frac{dx}{x-2} + 5 \int \frac{dx}{x-1} = 9 \ln|x-2| + 5 \ln|x-1|$$

$$\int \frac{1}{x-1} + c = \frac{9}{5} \ln \left| \frac{x-2}{x-1} \right| + c$$

$$\text{d) } \frac{x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} \Leftrightarrow A(x+1) + B(x+3) = x$$

Multiply by lcm.

To find A: set $x = -3 \Rightarrow A(-3+1) + 0 = -3 \Rightarrow A = \frac{3}{2}$.

To find B: set $x = -1 \Rightarrow 0 + 2B = -1 \Rightarrow B = -\frac{1}{2}$

$$\begin{aligned}\therefore \int \frac{x}{x^2 + 4x + 3} &= \frac{2}{3} \int \frac{dx}{x+3} + -\frac{1}{2} \int \frac{dx}{x+1} \\ &= \frac{2}{3} \ln |x+3| - \frac{1}{2} \ln |x+1| + c\end{aligned}$$

$$e) \quad \frac{12}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \Rightarrow A(x+3) + B(x-3) = 12.$$

Multiplying by $x^2 - 9$. **To find A:** set $x = 3 \Rightarrow 6A = 12 \Rightarrow A = 2$.

To find B: set $x = -3 \Rightarrow 0 - 6B = 12 \Rightarrow B = -2$.

$$\begin{aligned}\therefore \int \frac{12}{x^2 - 9} dx &= \int \frac{2}{x-3} dx - \int \frac{2}{x+3} dx \\ &= 2 \ln |x-3| - 2 \ln |x+3| + c = 2 \ln \left| \frac{x-3}{x+3} \right| + c\end{aligned}$$

$$f) \int \frac{4x+5}{(x+2)(x-2)} dx \leftarrow \text{Exercise left for you.}$$

b) Integration of repeated linear factor

$$\int \frac{N(x)}{(ax+b)^k} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \dots + \frac{C}{(ax+b)^k}$$

Illustrative Example

$$20. \quad \text{Find: a) } \int \frac{2x+3}{x(x+1)^2} dx \quad \text{b) } \int \frac{x^2-2x-3}{x^2(x^2+5x+6)}$$

$$\text{Solution: } \frac{2x+3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Leftrightarrow A(x+1)^2 + Bx(x+1) + cx = 2x+3. \text{ To find A: set } x=0 \Rightarrow A(0+1)^2 + 0 + 0 = 0+3$$

$$\Rightarrow A = 3. \text{ To find C: set } x = -1$$

$$\Rightarrow 0 + 0 - c = 2(-1) + 3 \Rightarrow c = -1.$$

To find B: set any convenient number, let $x = 1$.

$$\Rightarrow A(1+1)^2 + B(2) + c = 2(1) + 3$$

$$\Rightarrow 4A + 2B + c = 5 \Rightarrow 12 + 2B - 1 = 5 \Rightarrow B = -3.$$

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$$\therefore \int \frac{2x+3}{x(x+1)^2} dx = \int \left(\frac{3}{x} - \frac{3}{x+1} + \frac{-1}{(x+1)^2} \right) dx$$

$$= 3 \ln |x| - 3 \ln |x+1| + \frac{1}{x+1} + c$$

$$b) \frac{x^2 - 2x - 3}{x^2(x+2)(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{D}{x+3}$$

$$\Rightarrow Ax(x+2)(x+3) + B(x+2)(x+3) + C(x^2)(x+3) + D(x^2)(x+2) = x^2 - 2x - 3.$$

To find B: set $x=0 \Rightarrow 6B = 0 - 0 - 3 \Rightarrow B = -\frac{1}{2}$

To find C: set $x=-2 \Rightarrow 0 + 0 + 4C + 0 = 4 + 4 - 3 = 5 \Rightarrow C = \frac{5}{4}$

To find D: set $x=-3 \Rightarrow 0 + 0 + 0 + 9D = 9 + 6 - 3 = 12$

$$\Rightarrow D = \frac{12}{9} = \frac{4}{3}$$

To find A: set any x , let $x = -1$

$$\Rightarrow -2A + 2B + 2C + D = 1 + 2 - 3 = 0.$$

$$\Rightarrow -2A + 2\left(-\frac{1}{2}\right) + 2\left(\frac{5}{4}\right) + \frac{4}{3} = 0 \Rightarrow A = \frac{17}{12}$$

$$\therefore \int \frac{x^2 - 2x - 3}{x^2(x+2)(x+3)} dx$$

$$= \frac{17}{12} \int \frac{dx}{x} - \frac{1}{2} \int \frac{dx}{x^2} + \frac{5}{4} \int \frac{dx}{x+2} + \frac{4}{3} \int \frac{dx}{x+3}$$

$$= \frac{17}{12} \ln |x| + \frac{1}{2x} + \frac{5}{4} \ln |x+2| + \frac{4}{3} \ln |x+3| + c$$

Improper rational expression

For $\frac{N(x)}{D(x)}$ if $N(x) \geq D(x)$, then we first perform long division.

Illustrative Example

21. Find: a) $\int \frac{x+4}{x-1} dx$ b) $\int \frac{x+2}{4x-5} dx$ c) $\int \frac{x}{3x+4} dx$
 d) $\int \frac{x^2-4}{x^2-9} dx$ e) $\int \frac{x^3}{x^2-3x+2} dx$ f) $\int \frac{x^2-x-2}{x^2-9} dx$

Solution: Since the degree of numerator is equal to degree of denominators so we perform division.

a) $\int \frac{x+4}{x-1} dx = \int \left(1 + \frac{5}{x-1} \right) dx = x + 5 \ln |x-1| + c \leftarrow \text{by long division.}$

b) $\int \frac{x+2}{4x-5} dx = \int \left(\frac{1}{4} + \frac{13}{4(4x-5)} \right) dx = \frac{1}{4}x + \frac{13}{16} \ln |4x-5| + c$

c) $\int \frac{x}{3x+4} dx = \int \left(\frac{1}{3} + \frac{-4}{3(3x+4)} \right) dx = \frac{1}{3}x - \frac{4}{9} \ln |3x+4| + c$

d) $\int \frac{x^2-4}{x^2-9} dx = \int \left(1 + \frac{5}{x^2-9} \right) dx$ and $\frac{5}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$
 $\hookrightarrow A(x+3) + B(x-3) = 5$, set $x=3$ and $-3 \Rightarrow A = \frac{5}{6}, B = -\frac{5}{6}$

$$\therefore \int \frac{x^2-4}{x^2-3x+2} dx = \int \left(1 + \frac{5}{6(x-3)} + \frac{-5}{6(x+3)} \right) dx$$

$$= x + \frac{5}{6} \ln |x-3| - \frac{5}{6} \ln |x+3| + c$$

e) $\int \frac{x^3}{x^2-3x+2} dx = \int \left(x+3 + \frac{7x-6}{(x-2)(x-1)} \right) dx$, again

$$\frac{7x-6}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Leftrightarrow A(x-1) + B(x-2) = 7x-6,$$

let $x=2 \Rightarrow A=8$ and let $x=1 \Rightarrow B=-1$.

$$\therefore \int \frac{x^3}{x^2-3x+2} dx = \int \left(x+3 + \frac{8}{x-2} + \frac{-1}{x-1} \right) dx = \frac{x^2}{2} + 8 \ln |x-2| - \ln |x-1| + c$$

Exercise left for you.

ii) Quadratic factors of the form $\frac{N(x)}{ax^2 + bx + c}$ with $b^2 - 4ac < 0$.

Use: $\frac{N(x)}{ax^2 + bx + c} = \frac{Bx + c}{ax^2 + bx + c}$

Illustrative Example

22. Find: a) $\int \frac{x}{x^4 - 1} dx$ b) $\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx$

Solution: $\frac{x}{x^4 - 1} = \frac{x}{(x^2 - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{Cx + D}{x^2 + 1}$

$\Rightarrow A(x^2 + 1)(x + 1) + B(x - 1)(x^2 + 1) + (Cx + D)(x - 1)(x + 1) = x$

To find A: set $x = 1 \Rightarrow 4A + 0 + 0 = 1 \Rightarrow A = \frac{1}{4}$

To find B: set $x = -1 \Rightarrow 0 - 4B + 0 = -1 \Rightarrow B = \frac{1}{4}$

To find D: set any x , let $x = 0 \Rightarrow A - B - D = 0 \Rightarrow$ since $A = \frac{1}{4}$, $B = \frac{1}{4}$, so that $\frac{1}{4} - \frac{1}{4} - D = 0 \Rightarrow D = 0$

To find C: set any x , other than, -1, 1 and 0 let $x = 2$.

$\Rightarrow 15A + 5B + (2C + D)3 = 2 \Rightarrow \frac{15}{4} + \frac{5}{4} + 6C + 0 = 2 \Rightarrow C = -\frac{1}{2}$

$$\begin{aligned} \therefore \int \frac{x}{x^4 - 1} dx &= \frac{1}{4} \int \frac{dx}{x - 1} + \frac{1}{4} \int \frac{dx}{x + 1} + -\frac{1}{2} \int \frac{x}{x^2 + 1} dx \\ &= \frac{1}{4} \ln \left| \frac{x^2 - 1}{x^2 + 1} \right| + c \end{aligned}$$

b) $\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2} \leftarrow \text{Multiplying by } (x^2 + 2)^2$

$\Rightarrow 8x^3 + 13x = (Ax + B)(x^2 + 2) + (Cx + D) \leftarrow \text{Expanding} \Rightarrow 8x^3$

$13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D \leftarrow \text{Equate} \Rightarrow A = 7, B = 0,$
 and $2Ax + Cx + 13x \Rightarrow 2A + c = 13 \Rightarrow 16 + c = 13 \Rightarrow c = -3$ and $D = 0$

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx = \int \frac{8x}{x^2 + 2} dx + \int \frac{-3x}{(x^2 + 2)} dx$$

$$= 4\ln(x^2 + 2) + \frac{3}{2(x^2 + 2)} + c$$

C. Integration by parts

Integration by parts is method which is the reverse part of the product rule of **differentiation**.

Let u and v be functions of $x \Rightarrow u = u(x)$ and $v = v(x)$.

Then $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$ solving for $u \frac{dv}{dx}$, gives

$$\Rightarrow u \frac{dv}{dx} = \frac{d}{dx}(uv) - v \frac{du}{dx} \leftarrow \text{integrating both side}$$

$$\Rightarrow \int u \frac{dv}{dx} (dx) = \int \frac{d}{dx}(uv) dx - \int v \frac{du}{dx} \leftarrow \text{so that yield}$$

$$\boxed{\int u dv = uv - \int v du}$$

• This formula express the original integral in terms of another integral.

Depending on the choices of u and dv it may be easier to evaluate the 2nd integral than the original one.

General approach to choose u and dv

i) $\int \ln x dx$, use $u = \ln x, dv = dx$

ii) $\int x^n \ln x dx$ use $u = \ln x, dv = x^n dx$

iii) $\int x^n e^x dx$ use $u = x^n, dv = e^x dx$

iv) $\int x^n \sin x dx$: use $u = x^n, dv = \sin x dx$

v) $\int e^x \sin x dx$: use $u = \sin x, dv = e^x dx$

i) **Type** $\int \ln x dx$; Let, $u = \ln x, dv = dx$

Illustrative Example

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23. Find a) $\int \ln x dx$ b) $\int \log_3^x dx$ c) $\int \log(4x+3) dx$

Solution: Use $\int u dv = uv - \int v du$ and choice $u = \ln x$, $dv = dx$

a) Let $u = \ln x \Rightarrow du = \frac{dx}{x}$ and $dv = dx \Rightarrow v = \int dx = x$

$$\therefore \int u dv = uv - \int v du \text{ gives } \int \ln x dx = x \ln x - \int dx = (x \ln x - x) + c$$

b) We have $\log_3^x = \frac{\ln x}{\ln 3}$, so that $\int \log_3^x dx = \int \frac{\ln x}{\ln 3} dx$

$$= \frac{1}{\ln 3} \int \ln x dx = \frac{1}{\ln 3} (x \ln x - x) + c$$

c) We have $\log(4x+3) =$

$$\frac{\ln(4x+3)}{\ln 10} \Rightarrow \int \log^{(4x+3)} dx = \int \frac{\ln(4x+3)}{\ln 10} dx$$

let $u = 4x+3 \Rightarrow du = 4dx \Rightarrow dx = \frac{du}{4}$, $\frac{1}{4 \ln 10} \int \ln u du$

$$= \frac{1}{4 \ln 10} (u \ln u - u) + c = \frac{1}{4 \ln 10} (4x+3) \ln(4x+3) + c$$

Note:

i) $\int \ln x dx = (x \ln x - x) + c$

ii) $\int \log_a^x dx = \frac{1}{\ln a} (x \ln x - x) + c$

ii) Type two $\int x^n \ln x dx$. For this type use $u = \ln x$, $dv = x^n$

24.

Find

Illustration Example

a. $\int x \ln x dx$

b. $\int x^2 \ln x dx$

c. $\int (\ln x)^2 dx$

d. $\int x^3 \ln x dx$

e. $\int x \ln(4x) dx$

f. $\int x \ln(nx) dx$

g. $\int x^n \ln x dx$ h. $\int x \log_4 x dx$ i. $\int x^2 \log_4 x dx$

Solution: Use $\int u dv = uv - \int v du$. Then a) let $u = \ln x$
 $u = \ln x$ and $dv = x dx$

$$\Rightarrow du = \frac{dx}{x} \text{ and } dv = x dx \Rightarrow v = \int x dx = \frac{x^2}{2}$$

$$\therefore \int u dv = uv - \int v du \text{ substituting}$$

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

b. Let $u = \ln x \Rightarrow du = \frac{dx}{x}$ and $dv = x^2 dx \Rightarrow v = \int x^2 dx = \frac{x^3}{3}$

$$\therefore \int u dv = uv - \int v du \text{ substitute}$$

$$\Rightarrow \int x^2 \ln x dx = \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + c$$

c. Let $u = (\ln x)^2 \Rightarrow du = \frac{2 \ln x}{x} dx$, $dv = dx \Rightarrow v = \int dx = x$. Then

$$\int u dv = uv - \int v du \text{ will be}$$

$$\begin{aligned} \int (\ln x)^2 dx &= x(\ln x)^2 - \int x \left(\frac{2 \ln x}{x} \right) dx \\ &= x(\ln x)^2 - 2(x \ln x) + 2x + c \end{aligned}$$

d. Let $u = \ln x \Rightarrow du = \frac{dx}{x}$, $dv = x^3 dx \Rightarrow v = \int x^3 dx = \frac{x^4}{4}$, thus

$$\int u dv = uv - \int v du \text{ gives}$$

$$\int x^3 \ln x dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \left(\frac{dx}{x} \right) = \frac{x^4}{4} \ln x - \frac{x^4}{16} + c$$

e. $\int x \ln(4x)$ Exercise left for you.

$$\text{Let } u = (\ln x) \Rightarrow du = \frac{1}{x} dx, dv = x dx \Rightarrow v = \int x dx = \frac{x^2}{2},$$

$$\text{then } \int u dv = uv - \int v du$$

$$\Rightarrow \int x \ln(\ln x) dx = \frac{x^2}{2} \ln(\ln x) - \int \frac{x^2}{2} \left(\frac{1}{x} \right) dx = \frac{x^2}{2} \ln(\ln x) - \frac{x^2}{4} + c$$

$$\text{Let } u = \ln x \Rightarrow du = \frac{1}{x} dx, dv = x^n dx$$

$$\Rightarrow v = \int x^n dx = \frac{x^{n+1}}{n+1}, \text{ then } \int u dv = uv - \int v du,$$

$$\Rightarrow \int x^n \ln x dx = \frac{x^{n+1}}{n+1} (\ln x) - \frac{1}{n+1}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + c, n \neq -1$$

Observe that:

$$\text{i) } \int x^3 \ln x dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + c$$

$$\text{ii) } \int x^5 \ln x dx = \frac{x^6}{6} \ln x - \frac{x^6}{36} + c$$

$$\begin{aligned} \text{iii) } \int x \log_3 x dx &= \int \frac{x \ln x}{\ln 3} dx = \frac{1}{\ln 3} \left(\frac{x^2}{2} (\ln x) - \frac{x^2}{4} \right) + c \\ &= \frac{x^2 \ln x}{2 \ln 3} - \frac{x^2}{4 \ln 3} + c \end{aligned}$$

$$\text{iv) } \int x^3 \log_4 x dx = \int \frac{x^3 \ln x}{\ln 4} dx = \frac{1}{\ln 4} \int x^3 \ln x dx = \frac{x^4 \ln x}{4 \ln 4} - \frac{x^4}{16 \ln 4} + c$$

iii) Part $\int x^n e^x dx$ and $\int x^n \cos x dx$

Illustrative Example

25. Find

a. $\int x e^{-x} dx$ b. $\int x \cos(nx) dx$ c. $\int x \sin(kx) dx, k > 0$

Solution: Let $u = x \Rightarrow du = dx$, $dv = e^x dx \Rightarrow v = \int e^x dx = e^x$.

Then $\int x \sin(kx) dx, k > 0$

$$\Rightarrow \int x e^{-x} dx = -x e^{-x} - \int -e^{-x} dx = -x e^{-x} - e^{-x} + c$$

b. Let $u = x \Rightarrow du = dx$ and $dv = \cos(nx) dx$

$$\Rightarrow v = \int \cos nx dx = \frac{\sin(nx)}{n}$$

$$\therefore \int u dv = uv - \int v du \Rightarrow \int x \cos(nx) dx$$

$$= \frac{x \sin(nx)}{n} - \int \frac{\sin(nx)}{n} dx = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} + c$$

Observe:

$$i) \int x \cos 2x dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$$

$$ii) \int x \cos(5x) dx = \frac{x \sin(5x)}{5} + \frac{\cos(5x)}{25} + c$$

c. Let $u = x \Rightarrow du = dx$, $dv = \sin$

$$(kx) dx \Rightarrow v = \int \sin(kx) dx = -\frac{\cos(kx)}{k}$$

$\therefore \int u dv = uv - \int v du$ is the same as

$$\int x \sin(kx) dx =$$

$$\frac{-x \cos(kx)}{k} - \int -\frac{\cos(kx)}{k} dx = \frac{-x \cos(kx)}{k} + \frac{\sin(kx)}{k^2} + c$$

Observe that:

$$i) \int x \sin(2x) dx = \frac{-x \cos 2x}{2} + \frac{\sin(2x)}{4} + c$$

$$\text{ii) } \int x \sin(3x) dx = \frac{-x \cos(3x)}{3} + \frac{\sin(3x)}{9} + c$$

Repeated use of integration by part

Illustrative Example

26. Find: a. $\int e^x \sin x dx$ b. $\int e^x \cos(3x) dx$ c. $\int x^2 e^x dx$
 d. $\int e^{3x} \sin x dx$

Solution: a. Let $u = \sin x \Rightarrow du = \cos x dx$, $dv = e^x dx$

$\Rightarrow v = \int e^x dx = e^x$. Then $\int u dv = uv - \int v du$ is the same as

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx.$$

Again apply integration by part for 2nd time.

$$\Rightarrow \int e^x \cos x dx, \text{ let } u = \cos x \Rightarrow du = -\sin x dx \text{ let } dv = e^x dx$$

$$\Rightarrow v = e^x. \text{ Then } \int e^x \cos x = e^x \cos x + \int e^x \sin x dx, \text{ collecting}$$

$$\Rightarrow \int e^x \sin x dx = e^x \sin x - (e^x \cos x + \int e^x \sin x dx)$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x dx \leftarrow \text{adding to the left part}$$

$$\Rightarrow 2 \int e^x \sin x dx = e^x \sin x - e^x \cos x + c$$

$$\Rightarrow \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + c$$

b. $u = \cos 3x \Rightarrow du = -3 \sin 3x dx$, $dv = e^x dx \Rightarrow v = e^x$ then

$$\int u dv = uv - \int v du \Rightarrow \int e^x \cos(3x) dx$$

$$= e^x \cos 3x - \int e^x (-3) \sin 3x dx = e^x \cos 3x + 3 \int e^x \sin 3x dx$$

Again apply integration by part 2nd time for $\int e^x \sin 3x dx$.

Let $u = \sin 3x \Rightarrow du = 3 \cos 3x dx$, $dv = e^x dx \Rightarrow v = e^x$ then

$$\int u dv = uv - \int v du.$$

$\int e^x \sin 3x dx = e^x \sin 3x - 3 \int e^x \cos(3x) dx$. Thus collecting

$$\Rightarrow \int e^x \cos(3x) dx = e^x \cos 3x + 3 \int e^x \sin(3x) dx$$

$$= e^x \cos(3x) + 3 \left[e^x \sin(3x) - 3 \int e^x \cos(3x) dx \right]$$

$$\Rightarrow 10 \int e^x \cos(3x) dx = e^x \cos(3x) + 3e^x \sin(3x) \Rightarrow \int e^x \cos(3x) dx$$

$$= \frac{e^x (\cos 3x + 3 \sin(3x))}{10} + c$$

Note: (Keep in touch).

$$\text{i) } \int e^{ax} \sin(bx) dx = \frac{e^{ax} (\sin(bx) - b \cos(bx))}{a^2 + b^2} + c$$

$$\text{ii) } \int e^{ax} \cos(bx) dx = \frac{e^{ax} (a \cos(bx) + b \sin(bx))}{a^2 + b^2} + c$$

Thus, we can see that:

$$\text{i) } \int e^{3x} \sin x dx = \frac{e^{3x} (3 \sin x - \cos x)}{3^2 + 1} + c$$

$$= \frac{e^{3x} (3 \sin x - \cos x)}{10} + c$$

$$\text{ii) } \int e^{-x} \sin(2x) dx = e^{-x} \frac{-\sin(2x) - 2 \cos(2x)}{(-1)^2 + (2)^2} + c$$

$$= \frac{-e^{-x} (\sin 2x + 2 \cos(2x))}{5} + c$$

$$\text{iii) } \int e^{2x} \cos(3x) dx = \frac{e^{2x} 2 \cos(3x) + 3 \sin(3x)}{2^2 + 3^2} + c$$

$$= \frac{e^{2x} (2 \cos(3x) + 3 \sin(3x))}{13} + c$$

d. Tabular method

Integration of the form

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- $\int x^n e^{ax} dx$
- $\int x^n \sin(bx) dx$
- $\int x^n \cos(bx) dx$ we can find by tabular method

To find x^n differentiated until you obtain 0.

And e^{ax} , $\sin(bx)$, $\cos(bx)$ integrate repeatedly.

Illustrative Example

27. Find:

- a. $\int x^2 e^x dx$ b. $\int (x^2 + 4x) e^{3x} dx$ c. $\int x^3 \sin(2x) dx$
 d. $\int x^2 \cos(3x) dx$ e. $\int x^2 \sin x dx$ f. $\int x e^2 dx$

Solution:

Derivative of polynomial

Integral (repeated anti derivative)

x^2		e^x
$2x$	+	e^x since $\int e^x dx = e^x$
2	-	e^x
0	+	e^x

$$\therefore \int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + c$$

b. Repeated deriv.

Repeated anti derive. (1, 2, 3)

$x^2 + 4x$		e^{3x}
$2x + 4$	+	$\frac{e^{3x}}{3}$ since $\int e^{3x} dx = \frac{e^{3x}}{3}$
2	-	$\frac{e^{3x}}{9}$ since $\int \frac{e^{3x}}{3} dx = \frac{e^{3x}}{9}$
0	+	$\frac{e^{3x}}{27}$ since $\int \frac{e^{3x}}{9} dx = \frac{e^{3x}}{27}$

$$\therefore \int (x^2 + 4x) e^{3x} dx = (x^2 + 4x) \frac{e^{3x}}{3} - (2x + 4) \frac{e^{3x}}{9} + 2 \frac{e^{3x}}{27} + c$$

c. Exercise left for you.

$$d. \int x^2 \cos(3x) dx = \frac{x^2 \sin 3x}{3} + \frac{2x \cos 3x}{9} - \frac{2 \sin 3x}{27} + c$$

c. Exercise left for you.

1.3 e^{2-x} since $\int e^{2-x} dx = -e^{2-x} dx = -e^{2-x}$

1 $-e^{2-x}$ since $-\int e^{2-x} dx = e^{2-x}$

0 e^{2-x}

$$\therefore \int x e^{2-x} dx = -x e^{2-x} - e^{2-x} + c$$

Solved problem

28. $\int e^{\sqrt{x}} dx$ is equal to:

A. $2\sqrt{2x} - 2e^{\sqrt{x}} + c$ B. $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$

C. $\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}} + c$ D. none

Solution: Let $y = \sqrt{x} \Rightarrow dy = \frac{dx}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x} dy$

$\Rightarrow dx = 2y dy$, since $y = \sqrt{x}$

$\Rightarrow \int e^{\sqrt{x}} dx = \int e^y (2y dy) = \int 2ye^y dy \leftarrow \text{Rewriting}$

Let $u = 2y \Rightarrow du = 2dy$ and $dv = e^y dy \Rightarrow v = e^y$

$\therefore \int u dv = uv - \int u du \Rightarrow \int 2ye^y dy = 2ye^y - 2 \int e^y dy$

$= 2ye^y - 2e^y + c \Rightarrow 2e^y (y - 1) + c = 2e^{\sqrt{x}} (\sqrt{x} - 1) + c.$

$\therefore$ **Answer: B**

29. $\int \frac{dx}{1+e^x}$ is equal to:

A. $\frac{1}{2}x - \frac{1}{2}\ln(e^x + 1) + c$

C. $\frac{1}{2}\ln(e^x + 1) + c$

B. $x - \ln(e^x + c) + c$

D. $x + \frac{1}{2}\ln(e^x + c) + c$

Solution: Let $u = 1 + e^x \Rightarrow du = e^x dx \Rightarrow dx = \frac{du}{e^x}$, since $e^x = u - 1$

$$\Rightarrow \int \frac{du}{u(u-1)} = \int \left(\frac{A}{u} + \frac{B}{u-1} \right) du, \text{ since } A = -1 \text{ and } B = 1$$

$$\Rightarrow \int \frac{-1}{u} du + \int \frac{1}{u-1} du = -\ln u + \ln(u-1) + c$$

$$= \ln \left(\frac{u-1}{u} \right) + c = \ln \left(\frac{e^x}{e^x + 1} \right) + c = x - \ln(e^x + 1) + c \therefore$$

Answer: B

30. $\int x^3 e^{x^2} dx$ is equal to:

A. $\frac{1}{2} e^{x^2} (x^2 + 1) + c$

C. $\frac{1}{2} x^2 (e^{x^2} + 1) + c$

B. $\frac{1}{2} e^{x^2} (x^2 - 1) + c$

D. $\frac{1}{2} x^2 (e^{x^2} - 1) + c$

Solution: Let $u = x^2 \Rightarrow du = 2x dx \Rightarrow x dx = \frac{du}{2}$

$$\int x^3 e^{x^2} dx = \int x^2 e^{x^2} (x dx) = \int u e^u \frac{du}{2} = \frac{1}{2} \int u e^u du$$

Note: $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$

$$\therefore \frac{1}{2} \int u e^u du = \frac{1}{2} \left[u e^u - \int e^u du \right] = \frac{1}{2} u e^u - \frac{1}{2} e^u + c = \frac{1}{2} e^u (u - 1) + c$$

$$\Rightarrow \frac{1}{2} e^{x^2} (x^2 - 1) + c$$

$\therefore$ Answer: A

31. $\int \frac{\ln x}{\sqrt{x}} dx \dots$ (UEE)

A. $2\sqrt{x} \ln x + c$

C. $2\sqrt{x} (\ln x - 2)$

B. $\sqrt{x} \ln x - \sqrt{x} + c$

D. None

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Solution: Let $u = \ln x \Rightarrow du = \frac{dx}{x}$ and

$$v = x^{-\frac{1}{2}} dx \Rightarrow v = \int x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}}, \text{ thus}$$

$\int u dv = uv - \int v du$ is substitute

$$\int \frac{\ln x}{\sqrt{x}} dx = (2\sqrt{x}) \ln x - \int 2(x^{\frac{1}{2}}) \frac{dx}{x} = 2\sqrt{x} (\ln x - 4) + c$$

$\therefore$ Answer: C

32. $\int x \sin x dx \dots$ (UEE)

- A. $x^2 \cos x + c$ C. $x^2 \sin x - x \cos x + c$
 B. $\sin x - x \cos x + c$ D. $-\sin x + x^2 \cos x + c$

Solution: Let $u = x \Rightarrow du = dx$ and $dv = \sin x dx \Rightarrow v = -\cos x$

$$\therefore \int u dv = uv - \int v du = \int x \sin x dx = -x \cos x + \int \cos x dx$$

$$= \sin x - x \cos x + c$$

$\therefore$ Answer: B

33. $\int \left(\frac{x}{x+1} \right)^2 dx \dots$ (UEE)

- A. $x + \ln |x+1| + c$ C. $x - \frac{1}{x+1} - 2 \ln |x+1| + c$
 B. $\frac{1}{3}x + \frac{1}{3} \ln |x+1| + c$ D. $x - \left(\frac{2}{x+1} \right)^2 - \frac{1}{x+1} + c$

Solution:

$$\int \left(\frac{x}{x+1} \right)^2 dx = \int \left(1 - \frac{1}{x+1} \right)^2 dx = \int \left(1 - \frac{2}{x+1} + \frac{1}{(x+1)^2} \right) dx$$

$$\Rightarrow x - 2 \ln |x+1| - \frac{1}{x+1} + c$$

$\therefore$ Answer: C

34. $\int \frac{x^2 - 1}{x^3 + x} dx \dots$ UEE 2011

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A. $\ln|x^2+1| + \ln|x| + c$ C. $\frac{1}{2}\ln(x^2+1) - \ln|x| + c$

B. $\ln(x^2+1) - \ln|x| + c$ D. $2\ln(x^2+1) - \ln|x| + c$

Solution: $\frac{x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow A(x^2+1) + (Bx+C)x = x^2-1$

$\Rightarrow Ax^2 + Bx^2 + Cx + A = x^2 - 1 \dots$ comparing coef. of like term $\Rightarrow A+B=1$ and $C=0$ and $A+B=1 \Rightarrow -1+B=1 \Rightarrow B=2$

$$\therefore \int \frac{x^2-1}{x^3+x} = \int \left(\frac{-1}{x} + \frac{2x}{x^2+1} \right) dx = \int \frac{-1}{x} dx + \int \frac{2x}{x^2+1} dx$$

$$= -\ln|x| + \int \frac{2x}{x^2+1} dx$$

Let $u = x^2 + 1 \Rightarrow du = 2x dx \Rightarrow \int \frac{du}{u} = \ln u = \ln(x^2 + 1)$

$\therefore \int \frac{x^2-1}{x^3+x} dx = \ln(x^2+1) - \ln|x| + c \quad \therefore \text{Answer: B}$

35. $\int \frac{x}{x^2+2x+1} dx \dots$

A. $\ln(x^2+2x+1) + c$ C. $\ln(x^2+2x+1) - \frac{1}{x-1} + c$

B. $\ln|x+1| + \frac{1}{x+1} + c$ D. $\ln(x^2+2x+1) - \frac{1}{x-1} + c$

Solution: $\frac{x}{x^2+2x+1} dx = \int \frac{1}{x+1} dx - \int \frac{1}{(x+1)^2} dx$

$= \ln|x+1| + \frac{1}{x+1} + c \quad \therefore \text{Answer: B}$

36. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx \dots$ UEE

A. $\sin \sqrt{x} + \cos \sqrt{x} + c$ C. $2 \sin \sqrt{x} + c$

B. $-\sqrt{x} \cos \sqrt{x} + c$

D. $\frac{1}{2} \sqrt{x} \sin x + c$

Solution: Let $u = \sqrt{x} \Rightarrow du = \frac{dx}{2\sqrt{x}} \Rightarrow 2du = \frac{dx}{\sqrt{x}}$

$\Rightarrow 2 \int \cos u du = 2 \sin u + c = 2 \sin \sqrt{x} + c \therefore \text{Answer: B}$

37. $\int \frac{e^{2x} - 4x}{xe^{2x}} dx \dots$ UEE 2004 /12

A. $\ln |x| + 2e^{-2x} + c$

C. $\ln |x| - 4e^{2x} + c$

B. $\ln |x| + 4e^{-2x} + c$

D. $-\frac{1}{x^2} + 2e^{-2x} + c$

Solution: $\int \frac{e^{2x} - 4x}{xe^{2x}} dx = \int \left(\frac{1}{x} - 4e^{-2x} \right) dx \leftarrow \text{simplifying}$

$\Rightarrow \ln |x| + \frac{4}{2} e^{-2x} + c = \ln |x| + 2e^{-2x} + c \therefore \text{Answer: A}$

38. $\int (2 + \sin^2 x) \cot x dx$ is equal to: UEE

A. $2 \csc x - \sec^2 x + c$

C. $2 \ln |\sin x - \cos^2 x| + c$

B. $2 \ln |\sin x| + \frac{1}{2} \sin^2 x + c$

D. $2 \csc x + \sin x \cos x + c$

Solution: $\int (2 + \sin^2 x) \frac{\cos x}{\sin x} dx = \int \left(\frac{2 \cos x}{\sin x} + \sin x \cos x \right) dx$

$\Rightarrow \int \frac{2 \cos x dx}{\sin x} + \int \sin x \cos x dx$

Let $u = \sin x \Rightarrow du = \cos x dx$

$\Rightarrow \int 2 \frac{du}{u} + \int u du = 2 \ln |u| + \frac{u^2}{2} + c$

$= 2 \ln |\sin x| + \frac{\sin^2 x}{2} + c$

$\therefore \text{Answer: B}$

39. $\int \frac{x}{1 + e^{-x^2}}$ is equal to:

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A. $-\frac{1}{2}(x^2 + \ln(e^{-x^2} + 1)) + c$ C. $-\frac{1}{2}(x + \ln(e^{-x^2} - 1)) + c$
 B. $\frac{(x^2 + \ln(e^{-x^2} + 1)) + c}{2}$ D. $x + \ln(e^{-x^2} + 1) + c$

Solution: Let $u = -x^2 \Rightarrow du = -2x dx \Rightarrow x dx = -\frac{1}{2} du$

$$\begin{aligned}\int \frac{x}{1+e^{-x^2}} dx &= -\frac{1}{2} \int \frac{1}{1+e^u} du = -\frac{1}{2} \int \left(\frac{1+e^u}{1+e^u} - \frac{e^u}{1+e^u} \right) du \\ &= -\frac{1}{2} \int \left(1 - \frac{e^u}{1+e^u} \right) du = -\frac{1}{2} (u - \ln(e^u + 1)) + c \\ &\Rightarrow -\frac{1}{2} (-x^2 - \ln(e^{-x^2} + 1)) + c = \frac{(x^2 + \ln(e^{-x^2} + 1)) + c}{2}\end{aligned}$$

Answer: B

40. $\int \frac{\ln x}{x^2} dx$ is equal to:

A. $\frac{\ln x}{x} - \frac{1}{x} + c$ B. $-\frac{\ln x}{x} - \frac{1}{x} + c$ C. $-\frac{\ln x}{x} + \frac{1}{x} + c$

Solution: It requires integration by parts

let $u = \ln x \Rightarrow du = \frac{1}{x} dx$ and $dv = \frac{1}{x^2} dx \Rightarrow v = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2} dx = \frac{-\ln x}{x} + \int -\frac{1}{x^2} dx = \frac{-\ln x}{x} + \frac{1}{x} + c \therefore \text{Answer: C}$$

41. $\int \frac{e^x}{(e^x - 1)(e^x + 4)} dx$ is equal to:

A. $\frac{1}{5} \ln \left(\frac{e^x - 1}{e^x + 4} \right) + c$ C. $\frac{1}{5} \ln \left(\frac{e^x + 2}{e^x + 1} \right) + c$
 B. $\frac{1}{5} \ln \left(\frac{e^x + 2}{e^x + 1} \right) + c$ D. None

Solution: Let $y = e^x - 1 \Rightarrow e^x = y + 1$ and $dy = e^x dx$

$$\Rightarrow \int \frac{1}{y(y+5)} dy = \int \left(\frac{A}{y} + \frac{B}{y+5} \right) dy \Leftrightarrow A(y+5) + By = 1$$

$$\text{Let } y = 0 \Rightarrow A(0+5) = 1 \Rightarrow A = \frac{1}{5}$$

$$\text{Let } y = -5 \Rightarrow A(0) + B(-5) = 1 \Rightarrow B = -\frac{1}{5}$$

$$\therefore \int \frac{1}{5y} dy + \frac{-1}{5} \int \frac{dy}{y+5} \Rightarrow \frac{1}{5} \ln y - \frac{1}{5} \ln(y+5) = \frac{1}{5} \ln \left(\frac{y}{y+5} \right)$$

$$\text{but } y = e^x - 1, \text{ so that } \frac{1}{5} \ln \left(\frac{e^x - 1}{e^x + 4} \right) + c$$

$\therefore$ Answer: A

- Definite integrals
- Area of the curve
- Fundamental theorem of calculus

• Definite integral as area of the region

The area under a curve can also be found by integration. In geometry we have formula to find the area of figures such as **triangle, circle, rectangle and trapezoid**.

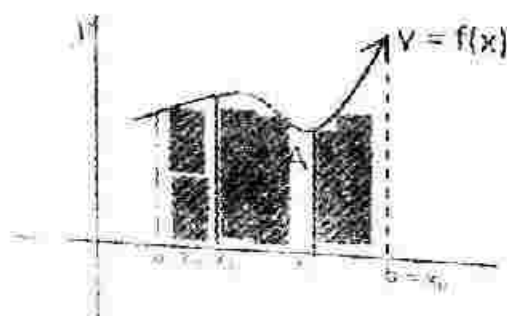
Now here, we going to determine *the area of the region (R)* by the curve $y = f(x)$ which is continuous on $[a, b]$ above the x -axis and between the line $x = a$ and $x = b$ calculated as follow.

To estimate the area, we will start from the definition of the area of rectangle.

- Define the area of the region divide into thinner and thinner rectangle.
- Divide the interval $[a, b]$ into n equal sub interval, these are: $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ with $x_0 = a$ and $x_n = b$

The length of each thinner rectangle is $x_1 - x_0 = x_2 - x_1 = \dots$
 $x_n - x_{n-1} = \Delta x$. Thus the region is divided into n stripes which is nearly

n rectangle of the same width. $\Delta x = \frac{b-a}{n}$



$$\Rightarrow b - a = n \Delta x$$

$$\Rightarrow b = a + n \Delta x$$

$$\Rightarrow x_i = a + i \Delta x$$

So that,

Area of rectangle = (length)(height).

Thus, $A_i = f(x_i)\Delta x$ ← area of i th rectangle,
 Δx ← length of rectangle
 $f(x_i)$ ← height i th rectangle,

Total area of the region is given by:

$$A = \sum_{i=0}^n A_i = A_0 + A_1 + A_2 + \dots$$

$$\begin{aligned} A_n &= f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_n)\Delta x \\ &= \Delta x [f(x_0) + f(x_1) + f(x_2) + \dots + f(x_n)] \\ &= \left[\sum_{i=0}^n f(x_i) \right] \Delta x = \frac{b-a}{n} \left[\sum_{i=0}^n f(x_i) \right] \end{aligned}$$

As n gets larger and larger without bound the area of the region become exact and express as:

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n f(x_i)$$

The definite integral

There is surprising link between the **definite integrals** and the **area** of the curve $y = f(x)$.

The definite integral of function f from a to b is the number

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x = \int_a^b f(x)dx = F(b) - F(a)$$

Where:

- $f(x)$ is continuous on $[a, b]$
- a and b are called limit of integration
- a is called the lower limit of integration
- b is called the upper limit of integration

To illuminate the above let we have illustrative example.

Illustrative Example

42. Divide the interval $[1, 3]$ into equal length of:

a. 3 - sub interval b. 5 - sub interval c. n - sub interval

Solution: a) We have $\Delta x = \frac{b-a}{n} = \frac{3-1}{3} = \frac{2}{3}$. Then we use,

$x_i = a + i \Delta x$, where $a = 1$, $b = 3$, $\Delta x = \frac{2}{3}$, $x_0 = 1 + 0 = 1$ and

$$x_1 = 1 + \frac{2}{3} = \frac{5}{3} \Rightarrow \left[1, \frac{5}{3}\right] \Rightarrow x_2 = 1 + 2\left[\frac{2}{3}\right] = \frac{7}{3} \leftarrow \text{add } \frac{2}{3} \text{ to}$$

each next element. The interval, $[x_0, x_1]$, $[x_1, x_2]$, and $[x_2, x_3]$

$$\Rightarrow \left[1, \frac{5}{3}\right], \left[\frac{5}{3}, \frac{7}{3}\right], \left[\frac{7}{3}, \frac{9}{3}\right] \leftarrow 3 \text{ sub equal interval.}$$

b. $[1, 3]$ into 5 - sub interval $\Delta x = \frac{b-a}{n} = \frac{3-1}{5} = \frac{2}{5}$.

$$\begin{aligned} & \left[1, 1 + \frac{2}{5}\right], \left[\frac{7}{5}, \frac{2}{5} + \frac{7}{5}\right], \left[\frac{9}{5}, \frac{2}{5} + \frac{9}{5}\right], \left[\frac{11}{5}, \frac{2}{5} + \frac{11}{5}\right], \left[\frac{13}{5}, \frac{2}{5} + \frac{13}{5}\right] \\ & = \left[1, \frac{7}{5}\right], \left[\frac{7}{5}, \frac{9}{5}\right], \left[\frac{9}{5}, \frac{11}{5}\right], \left[\frac{11}{5}, \frac{13}{5}\right] \text{ and } \left[\frac{13}{5}, 3\right] \end{aligned}$$

c. $[1, 3]$ in to n - sub equal interval.

$$\Rightarrow \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$$\Rightarrow \left[1, \frac{n+2}{n}\right], \left[\frac{n+2}{n}, \frac{n+4}{n}\right], \dots, \left[\frac{3n-2}{n}, 3\right] \leftarrow n \text{ sub interval.}$$

43. Estimate the area using the limit sum of rectangle of the graph $f(x) = x^2$ between the line $x = 1$ and $x = 2$ by dividing the interval $[1, 2]$ into equal length of

a. 3 - sub interval b. 6 - sub interval c. 10 - sub interval

Solution: We have area (A) = (height) (length)

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} = \frac{1}{3} \leftarrow n = 3.$$

$A = \sum_{i=1}^n A_i = \sum_{i=1}^n f(x_i) \Delta x \leftarrow f(x_i)$ the height of the i th rectangle. But

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} \text{ and } b = a + n \Delta x$$

$$\Rightarrow x_i = a + i \Delta x = 1 + \frac{i}{n}$$

$$\Rightarrow f(x_i) = f\left(1 + \frac{i}{n}\right) = \left(1 + \frac{i}{n}\right)^2 = 1 + \frac{2i}{n} + \frac{i^2}{n^2} \leftarrow \text{since } f(x) = x^2$$

$$\begin{aligned} A_n &= \sum_{i=1}^n A_i = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right) \left(\frac{1}{n}\right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right) \end{aligned}$$

$$\Rightarrow A_n = \sum_{i=1}^n A_i = \frac{1}{n} \left(\sum_{i=1}^n 1 + \frac{2}{n} \sum_{i=1}^n i + \frac{1}{n^2} \sum_{i=1}^n i^2 \right) \leftarrow \Sigma, \text{ rule } \dots (*)$$

Since $\sum_{i=1}^n 1 = n$ and $\sum_{i=1}^n i = \frac{1}{2} n(n+1) = \frac{n^2 + n}{2}$ also

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{2n^3 + 3n^2 + n}{6}; \text{ Then substitute (i) in } (*)$$

$$\Rightarrow A_n = \frac{1}{n} \left[n + \frac{2}{n} \left(\frac{n^2 + n}{2} \right) + \frac{1}{n^2} \left(\frac{2n^3 + 3n^2 + n}{6} \right) \right] = \frac{7}{3} + \frac{1}{n} + \frac{1}{2n} + \frac{1}{n^2}$$

$$\therefore A = \frac{7}{3} + \frac{1}{n} + \frac{1}{2n} + \frac{1}{n^2} \leftarrow \text{Area of } n\text{th sub-interval}$$

a. $n = 3$ - sub interval $\Rightarrow A = \frac{7}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{53}{18} \approx 2.94$

$$b. \quad n = 6 - \text{sub interval} \Rightarrow A = \frac{7}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36} = \frac{94}{36} \approx 2.61$$

As the number of interval increase without bound then the area:

$$A = \lim_{n \rightarrow \infty} \left(\frac{7}{3} + \frac{1}{n} + \frac{1}{2n} + \frac{1}{n^2} \right) = \frac{7}{3} = \int_1^2 x^2 dx$$

Theorem: If the function f is continuous on $[a, b]$ then

$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$ exist. That is f is integrable so that continuity implies integrability.

Example:

- $\int_{-2}^2 \frac{1}{x^2 - 1} dx$ is not integrable because $f(x) = \frac{1}{x^2 - 1}$ is not cont. at $x = -1$ and $x = 1$.
- $\int_{-1}^3 \frac{1}{x - 2} dx$ is doesn't exist because not cont. at $x = 2$.

Fundamental theorem of Calculus

This theorem will relate the rectangle and anti-derivative methods for calculating areas, and evaluating definite integrals using anti derivatives.

Theorem Fundamental theorem of calculus I.

If f is continuous on $[a, b]$ and F is an anti-derivative of f on $[a, b]$ then $\int_a^b f(x) dx = F(b) - F(a)$, if $F'(x) = f(x)$ for all $x \in [a, b]$.

Illustrative Example

44. Evaluate each definite integral:

$$a. \int_1^2 (x^2 - 3) dx \quad b. \int_0^2 (2x^2 - 3x + 2) dx$$

Solution: a. $\int_1^2 (x^2 - 3) dx = \left[\frac{x^3}{3} - 3x \right] = \left(\frac{8}{3} - 6 \right) - \left(\frac{1}{3} - 3 \right) = -\frac{2}{3}$

b. $\int (2x^2 - 3x + 2) dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x \right]_0^2$

$$= \left(\frac{16}{3} - \frac{12}{2} + 4 \right) - (0 - 0 + 0) = \frac{10}{3}$$

Properties of Definite integrals

Let f and g be integrable on $[a, b]$ and $C \in [a, b]$:

1. $\int_a^a f(x) dx = 0$
2. $\int_a^b f(x) dx = -\int_b^a f(x) dx$
3. $\int_a^b kf(x) dx = k \int_a^b f(x) dx$
4. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$
5. If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x) dx \geq 0$

- If f is an even function, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
- If f is an odd function, then $\int_{-a}^a f(x) dx = 0$.

Example: a. $\int_{-4}^4 x^2 dx = 2 \int_0^4 x^2 dx$

b. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx = 2 \int_0^{\frac{\pi}{2}} \cos x dx$ c. $\int_{-2}^2 x^3 dx = 0$

d. $\int_{-4}^4 (x^3 - x) dx = 0$

Illustrative Example

45. Evaluate the integrals

a. $\int_{-3}^2 |x+1| dx$ b. $\int_{-3}^4 |x^2 - 4| dx$ c. $\int_{-1}^2 |x^2 - x| dx$

Solution: a) since $|x+1| = x+1$ if $x \geq -1$ and $|x+1| = -x-1$ if $x < -1$

$$\Rightarrow \int_{-3}^2 |x+1| dx = \int_{-3}^{-1} (-x-1) dx + \int_{-1}^2 (x+1) dx$$

$$= \left(\frac{x^2}{2} + x \right) \Big|_{-3}^{-1} + \left(\frac{-x^2}{2} - x \right) \Big|_{-1}^2 = -2 + \frac{-3}{2} - \frac{7}{2}$$

b) Since $|x^2 - 4| = x^2 - 4$ if $x \leq -2$ or $x \geq 2$

$$\text{and } |x^2 - 4| = -x^2 + 4, \text{ if } -2 < x < 2. \Rightarrow \int_{-3}^4 |x^2 - 4| dx$$

$$= \int_{-3}^{-2} (x^2 - 4) dx + \int_{-2}^2 (-x^2 + 4) dx + \int_2^4 (x^2 - 4) dx$$

$$= \left(\frac{x^3}{3} - 4x \right) \Big|_{-3}^{-2} + \left(-\frac{x^3}{3} + 4x \right) \Big|_{-2}^2 + \left(\frac{x^3}{3} - 4x \right) \Big|_2^4$$

$$= F(2) - F(-3) + G(2) - G(-2) + R(4) - R(2) = 13.$$

c) Exercise left for you. Answer Hint

$$\int_{-1}^2 |x^2 - x| dx = \int_{-1}^0 (x^2 - x) dx + \int_0^1 (-x^2 + x) dx + \int_1^2 (x^2 - x) dx$$

46. Evaluate the integrals

a. If $f(x) = \begin{cases} x^2 + 2, & x < 2 \\ 5x - 4, & x \geq 2 \end{cases}$ then find $\int_{-1}^3 f(x) dx$

b. If $f(x) = \begin{cases} x^2 + 4, & x < 2 \\ 6x - 4, & 2 \leq x \leq 3 \\ 2x + 8, & x > 3 \end{cases}$ then $\int_0^4 f(x) dx$

Solution: a) $\int_{-1}^3 f(x) dx = \int_{-1}^2 (x^2 + 2) dx + \int_2^3 (5x - 4) dx$
 $= \left(\frac{x^3}{3} + 2x \right) \Big|_{-1}^2 + \left(\frac{5x^2}{2} - 4x \right) \Big|_2^3$

$$F(2) - F(-1) + G(3) - G(2) = \frac{35}{2}$$

b) $\int_0^4 f(x) dx = \int_0^2 (x^2 + 4) dx + \int_2^3 (6x - 4) dx + \int_3^4 (2x + 8) dx$

$$= \left(\frac{x^3}{3} + 4x \right) \Big|_0^2 + (3x^2 - 4x) \Big|_2^3 + (x^2 + 8x) \Big|_3^4$$

$$= F(2) - F(0) + G(3) - G(2) + R(4) - R(3) = \frac{110}{3}$$

Change of variables for Definite integrals

If $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Illustrative Example

1. Find

a. $\int_{-1}^1 \frac{2x+1}{e^{x^2+x}} dx$

b. $\int_{\frac{\pi^2}{4}}^{\pi} \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

c. $\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx$

d. $\int_1^2 x^2(x^3-3)^3 dx$

e. $\int_0^{\pi} \sin^2 x dx$

f. $\int_{-1}^0 \frac{x+2}{(x^2+4x-1)^{\frac{3}{2}}} dx$

g. $\int_{-1}^1 \frac{1}{1+e^{-x}} dx$

h. $\int_{\frac{\pi}{2}}^{\pi} x \sin x dx$

i. $\int_e^{e^2} x \ln x dx$

Solution: Let $u = x^2 + x \Rightarrow du = (2x+1) dx$ so that when $x = 1 \Rightarrow u = 2$ and $x = -1 \Rightarrow u = 0$.

$$\therefore \int_{-1}^1 \frac{2x+1}{e^{x^2+x}} dx = \int_0^2 \frac{du}{e^u} = -e^{-u} \Big|_0^2 = -e^{-2} - (-1) = 1 - e^{-2}$$

b) Let $u = \sqrt{x} \Rightarrow du = \frac{dx}{2\sqrt{x}}$ so that when $x = \pi^2 \Rightarrow u = \pi$ and

$$x = \frac{\pi^2}{4} \Rightarrow u = \frac{\pi}{2}$$

$$\begin{aligned}\therefore \int_{\frac{\pi}{2}}^{\pi} \frac{\cos \sqrt{x}}{\sqrt{x}} dx &= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \cos u du \\ &= \frac{\sin u}{2} \Big|_{\frac{\pi}{2}}^{\pi} = \frac{\sin \pi - \sin \left(\frac{\pi}{2}\right)}{2} = -\frac{1}{2}\end{aligned}$$

c) Let $u = \sqrt{1+\sqrt{x}} \Rightarrow u^2 = 1 + \sqrt{x} \Rightarrow \sqrt{x} = u^2 - 1$

$$\Rightarrow 2u du = \frac{dx}{2\sqrt{x}} \Rightarrow dx = 4\sqrt{x} u du = 4(u^2 - 1)u du \text{ so that when}$$

$$x=4 \Rightarrow u = \sqrt{3} \text{ and when } x=1 \Rightarrow u = \sqrt{2}.$$

$$\begin{aligned}\therefore \int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx &= \int_{\sqrt{2}}^{\sqrt{3}} \frac{4u^2(u^2-1)}{u^2-1} du = 4 \int_{\sqrt{2}}^{\sqrt{3}} u^2 du \\ &= \frac{u^3}{3} \Big|_{\sqrt{2}}^{\sqrt{3}} = \frac{9\sqrt{3} - 2\sqrt{2}}{3}\end{aligned}$$

d) Let $u = x^3 - 3 \Rightarrow du = 3x^2 dx \Rightarrow x^2 dx = \frac{du}{3}$. So that when $x=2 \Rightarrow u=5$ and when $x=1 \Rightarrow u=-2$.

$$\therefore \int_1^2 (x^3 - 3)^3 x^2 dx = \int_{-2}^5 u^3 \frac{du}{3} = \frac{1}{3} \left[\frac{u^4}{4} \right]_{-2}^5 = \frac{u^4}{12} \Big|_{-2}^5 = \frac{609}{12}$$

e) **Recall:** $\cos(2x) = \cos(x+x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$$\Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

$$\therefore \int_0^{\pi} \sin^2 x dx = \int_0^{\pi} \frac{1}{2}(1 - \cos 2x) = \left(\frac{1}{2}x - \frac{\cos 2x}{4} \right) \Big|_0^{\pi} = \frac{\pi}{2}$$

f) Let $u = x^2 + 4x - 1 \Rightarrow du = 2(x+2) dx \Rightarrow \frac{du}{2} = (x+2) dx$

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$$\int_{-1}^0 \frac{x+2}{(x^2+4x-1)} dx = \frac{1}{2} \int_{-4}^{-1} \frac{1}{4} du = -\frac{1}{2} \frac{u^{-3}}{3} \Big|_{-4}^{-1} = \frac{65}{384}$$

g) Since $\int \frac{1}{1+e^{-x}} dx = \int \frac{1}{1+\frac{1}{e^x}} dx = \int \frac{1}{\frac{e^x+1}{e^x}} dx = \int \frac{e^x dx}{1+e^x}$

$$\therefore \int_{-1}^1 \frac{1}{1+e^{-x}} dx = \int_{-1}^1 \frac{e^x}{1+e^x} dx = \ln(1+e^x) \Big|_{-1}^1$$

$$= \ln(1+e) - \ln\left(\frac{1+e}{e}\right) = 1$$

h) Using integration by parts: Let $u = x \Rightarrow du = dx$ and $dv = \sin x dx \Rightarrow v = -\cos x$.

$$\int u dv = uv - \int v du \Rightarrow \int x \sin x dx = -\cos x + \int \cos x dx$$

$$\therefore \int_{\frac{\pi}{2}}^{\pi} x \sin x dx = (-x \cos x) \Big|_{\frac{\pi}{2}}^{\pi} + (\sin x) \Big|_{\frac{\pi}{2}}^{\pi} = \pi(0) + (0-1) = \pi - 1$$

i) Using integration by parts: Let $u = \ln x \Rightarrow du = \frac{dx}{x}$ and $dv = x dx$

$$\Rightarrow x dx \Rightarrow v = \frac{x^2}{2} \text{ so that } \int u dv = uv - \int v du$$

$$\Rightarrow \int x \ln x dx = \frac{x^2 \ln x}{2} - \frac{1}{2} \int x dx$$

$$\begin{aligned} \therefore \int_e^{e^3} x \ln x dx &= \frac{x^2 \ln x}{2} \Big|_e^{e^3} - \left(\frac{x^2}{4} \right) \Big|_e^{e^3} \\ &= \left(\frac{3e^6}{2} - \frac{e^2}{2} \right) - \left(\frac{e^6}{4} - \frac{e^2}{4} \right) = \frac{5e^6 - e^2}{4} \end{aligned}$$

48. If $\int_1^7 f(x) dx = 4$ then find

$$\text{a) } \int_0^2 f(3x+1)dx \quad \text{b) } \int_{\sqrt{3}}^3 f(x^2-2)dx$$

Solution: Note: $\int_1^7 f(x)dx = F(7) - F(1) = 4$, so that:

a) Let $u = 3x + 1 \Rightarrow du = 3 dx \Rightarrow dx = \frac{du}{3}$. Then

$$\therefore \int_0^2 f(3x+1)dx = \int_1^7 f(u) \frac{du}{3} = \frac{1}{3} [F(7) - F(1)] = \frac{1}{3} (4) = \frac{4}{3}$$

b) Let $u = x^2 - 2 \Rightarrow du = 2x dx \Rightarrow x dx = \frac{du}{2}$ and when $x = 3 \Rightarrow u = 7$ and

when $x = \sqrt{3} \Rightarrow u = 1$.

$$\therefore \int_{\sqrt{3}}^3 f(x^2-2)x dx = \int_1^7 f(u) \frac{du}{2} = \frac{1}{2} [F(7) - F(1)] = \frac{4}{2} = 2$$

Solved Problem

49. $\int_0^2 x^2 \ln(1+x^3)dx \dots$ UEE

A. $\frac{2}{3} \ln 2 - \frac{2}{3}$ B. $6 \ln 3 - \frac{10}{3}$ C. $6 \ln 3 - \frac{8}{3}$ D. $18 \ln 3 - 24$

Solution: Let $u = 1 + x^3 \Rightarrow du = 3x^2 dx \Rightarrow x^2 dx = \frac{du}{3}$ and when $x = 2 \Rightarrow$

$u = 9$ and when $x = 0 \Rightarrow u = 1$.

$$\therefore \int_0^2 x^2 \ln(1+x^3)dx = \int_1^9 \ln u \frac{du}{3} = \frac{1}{3} (u \ln u - u) \Big|_1^9$$

$$= \frac{1}{3} ((9) \ln 9 - 9) - \frac{1}{3} (0 - 1) = 6 \ln 3 - \frac{8}{3} \quad \therefore \text{Answer: C}$$

50. If f' is continuous on $[0, 1]$ such that $f(0) = 3$ and $f(1) = -1$ then

$\int_0^1 f'(x) \sqrt{1+f(x)} dx \dots$ UEE

A. -12 B. $\frac{14}{3}$ C. $\frac{4\sqrt{2}-2}{3}$ D. $\frac{-16}{3}$

Let $u = 1 - f(x) \Rightarrow du = -f'(x) dx$ and when $x = 1$
 $u = 1 - f(1) = 0$ and $x = 0 \Rightarrow u = 1 - f(0) = 4$

$$\int_0^1 \sqrt{1-f(x)} dx = \int_4^0 \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} \Big|_4^0 = 0 - \frac{16}{3} = -\frac{16}{3}$$

cont. D

$$\int_0^1 (1 + e^{2x}) dx = \dots$$

A. $e - \frac{1}{e}$ B. $\frac{1}{e} - e$ C. $e - \frac{1}{e} - 1$ D. $e - \frac{1}{e}$

$$\int_0^1 e^{-x} (1 + e^{2x}) dx = \int_0^1 (e^{-x} + e^x) dx = -e^{-x} + e^x \Big|_0^1$$

$$= (-e^{-1} + e) - (-1 + 1) = e - \frac{1}{e} \therefore \text{Answer: A}$$

$\int_0^1 (x + 1)e^x dx$: A. 0 B. 1 C. e D. $\frac{e-1}{e^2}$

Solution: $u = x + 1 \Rightarrow du = dx$ and $dv = e^x dx \Rightarrow v = e^x$

$$\int_0^1 (x + 1)e^x dx = \int_0^1 (x + 1)e^x dx = (x + 1)e^x - \int e^x dx$$

$$= ((x + 1)e^x - e^x) \Big|_0^1 = (2e - e) - (1 - 1) = e \therefore \text{Answer: C}$$

$\int_0^1 \frac{3x^2 + 1}{e^{x^3 + x}} dx$: A. -2 B. $1 - e^{-2}$ C. e^{-2} D. $\frac{2}{e^2}$

Solution: Let $u = x^3 + x \Rightarrow du = (3x^2 + 1) dx$ when $x = 1 \Rightarrow u = 2$
 and $x = 0 \Rightarrow u = 0$

$$\int_0^2 \frac{du}{e^u} = -e^{-u} \Big|_0^2 = -e^{-2} - (-1) = 1 - e^{-2}$$

$\therefore \text{Answer: B}$

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A. $\ln(1 + e)$ B. $\ln(1 + e) - \ln 2$ C. $\ln(2 + e)$ D. $\ln 2$

Solution: Let $u = 1 + f(x) \Rightarrow du = f'(x) dx$ and when $x = 1$
 $\Rightarrow u = 1 + f(1) = 0$ and $x = 0 \Rightarrow u = 1 + f(0) = 4$

$$\therefore \int_0^1 f'(x) \sqrt{1+f(x)} dx = \int_4^0 \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} \Big|_4^0 = 0 - \frac{16}{3} = -\frac{16}{3}$$

$\therefore$ Answer: D

51. $\int_0^1 e^{-x}(1+e^{2x}) dx \dots$

A. $e - \frac{1}{e}$ B. $\frac{1}{e} - e$ C. $e - \frac{1}{e} - 1$ D. $e + \frac{1}{e}$

Solution: $\int_0^1 e^{-x}(1+e^{2x}) dx = \int_0^1 (e^{-x} + e^x) dx = -e^{-x} + e^x \Big|_0^1$

$$= (-e^{-1} + e) - (-1 + 1) = e - \frac{1}{e} \therefore \text{Answer: A}$$

52. $\int_0^1 (x+1)e^x dx$: A. 0 B. 1 C. e D. $\frac{e-1}{e^2}$

Solution: $u = x+1 \Rightarrow du = dx$ and $dv = e^x dx \Rightarrow v = e^x$

$$\therefore \int u dv = uv - \int v du \Rightarrow \int_0^1 (x+1)e^x dx = (x+1)e^x - \int e^x dx$$

$$= ((x+1)e^x - e^x) \Big|_0^1 = (2e - e) - (1 - 1) = e \therefore \text{Answer: C}$$

53. $\int_0^1 \frac{3x^2+1}{e^{x^3}+x} dx$: A. -2 B. $1 - e^{-2}$ C. e^{-2} D. $\frac{2}{e^2}$

Solution: Let $u = x^3 + x \Rightarrow du = (3x^2 + 1) dx$ when $x = 1 \Rightarrow u = 2$

and $x = 0 \Rightarrow u = 0$. $\int_0^2 \frac{du}{e^u} = -e^{-u} \Big|_0^2 = -e^{-2} - (-1) = 1 - e^{-2}$

$\therefore$ Answer: B

54. $\int_0^1 \frac{1}{e^{-x}(1+e^x)} dx \dots$ UEE 2003/11

A. $\ln(e-1)$ B. $\ln(1+e) - \ln 2$ C. $\ln(2+e)$ D. $\ln 2$

Solution: $\int_0^1 \frac{1}{e^{-x}(1+e^x)} dx = \int_0^1 \frac{e^x}{1+e^x} dx = \ln(1+e^x) \Big|_0^1$
 $= \ln(1+e) - \ln(2)$ **∴ Answer: B**

55. $\int_1^3 \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} dx$ is equal to:

A. $-\frac{\sqrt{3}}{4\pi}$ B. $-\frac{\sqrt{3}}{3\pi}$ C. $\frac{\sqrt{3}}{\pi}$ D. $-\frac{\sqrt{3}}{2\pi}$

Solution: $u = \frac{\pi}{x} \Rightarrow du = -\frac{\pi}{x^2} dx \Rightarrow \frac{-du}{\pi} = \frac{dx}{x^2}$,

and when $x = 1 \Rightarrow u = \pi$ and $x = 3 \Rightarrow u = \frac{\pi}{3}$.

$\therefore \int_{\frac{\pi}{3}}^{\pi} \cos u \frac{du}{3} = -\frac{\sin u}{\pi} \Big|_{\frac{\pi}{3}}^{\pi} = \frac{-1}{\pi} \left[\frac{\sqrt{3}}{2} - 0 \right] = -\frac{\sqrt{3}}{2\pi}$ **∴ Answer: D**

56. If $f(x) = \begin{cases} \sin x, & x \leq 0 \\ 3x\sqrt{x^2+1}, & x > 0 \end{cases}$ then $\int_{-\frac{\pi}{2}}^{\sqrt{8}} f(x) dx \dots$ UEE 2004

A. 17 B. 25 C. 27 D. $3\sqrt{8}$

Solution:

$$\int_{-\frac{\pi}{2}}^0 f(x) dx + \int_0^{\sqrt{8}} f(x) dx = \int_{-\frac{\pi}{2}}^0 \sin x dx + \int_0^{\sqrt{8}} 3x\sqrt{x^2+1} dx$$

Let $u = x^2 + 1 \Rightarrow du = 2x dx \Rightarrow x dx = \frac{du}{2}$, when $x = \sqrt{8}$

$\Rightarrow u = 9$ and $x = 0 \Rightarrow u = 1$. Hence

$$\int_0^{\sqrt{8}} 3x\sqrt{x^2+1} dx = \int_1^9 \frac{3}{2} \sqrt{u} du = u^{\frac{3}{2}} \Big|_1^9 = 27 - 1 = 26$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^0 \sin x dx + \int_0^{\sqrt{8}} 3x\sqrt{x^2+1} dx = -\cos x \Big|_{-\frac{\pi}{2}}^0$$

$$= -1 + 26 = 25$$

∴ Answer: B

57. $\int_1^2 x \left(\frac{1}{x^2} + 2 \ln x \right) dx$ is equal to:

A. $2 \ln 2 - 1$ B. $5 \ln 2 + \frac{5}{2}$ C. $5 \ln 2 + \frac{5}{2}$ D. $5 \ln 2 - \frac{3}{2}$

Solution: Implies $\int_1^2 \frac{dx}{x} + 2 \int_1^2 \frac{dx}{x} + 2 \int_1^2 x \ln x dx \leftarrow$ integration by

parts $\left[\ln x + x^2 \ln x - \frac{x^2}{2} \right]_1^2 = \ln(2) + 4 \ln 2 - 2 - \left(0 - 0 - \frac{1}{2} \right)$

$$= 5 \ln 2 - \frac{3}{2}$$

58. If $f(1) = 3, f(0) = 2$, then $\int_0^1 \frac{f'(x)}{f(x)} dx$

A. $\ln 3$ B. $\ln 2$ C. $\ln \left(\frac{3}{2} \right)$ D. $\ln 6$

Solution: Let $u = f(x) \Rightarrow du = f'(x) dx = \ln 3 - \ln 2 = \ln \left(\frac{3}{2} \right)$

∴ Answer: C

59. If $f(0) = 5, f(1) = 3$, then $\int_0^1 e^x (f(x) + f'(x)) dx$ is

A. $3e - 5$ B. $2e$ C. $5 - e$ D. $-2 + e$

Solution:

$$\int_0^1 e^x (f(x) + f'(x)) dx = e^x f(x) \Big|_0^1 = e^1 f(1) - e^0 f(0) = 3e - 5$$

∴ Answer: A

Fundamental theorem of Calculus (2)

i) If the function F is defined on $[a, b]$ by $F(x) = \int_a^x f(t)dt$ then

$$F'(x) = f(x) \text{ for all } x \in [a, b]. \text{ That is } \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

Example

a. If $F(x) = \int_0^x \cos t dt$, then find $F'(x)$:

Solution:

$$\int_0^x \cos t dt = \sin t \Big|_0^x = \sin x$$

$$\therefore F'(x) = \frac{d}{dx} \int_0^x \cos t dx = \frac{d}{dx} \sin x = \cos x$$

b. If $F(x) = \int_{\frac{\pi}{2}}^{x^3} \cos t dx$, then find $F'(x)$

$$\text{Solution: } F(x) = \int_{\frac{\pi}{2}}^{x^3} \cos t dt = \sin t \Big|_{\frac{\pi}{2}}^{x^3} = \sin(x^3) - 1$$

$$\Rightarrow F'(x) = \cos(x^3)(3x^2) = 3x^2 \cos(x^3).$$

c. If $F(x) = \int_1^{x^2+1} \frac{1}{t} dt$ then $F'(x) \dots$ UEE

A. $\ln(x^2 + 1)$ B. $2x \ln(x^2 + 1)$ C. $\frac{2x}{x^2 + 1}$ D. $\frac{1}{x^2 + 1} + c$

$$\text{Solution: } F(x) = \int_1^{x^2+1} \frac{1}{t} dt = \ln t \Big|_1^{x^2+1} = \ln(x^2 + 1)$$

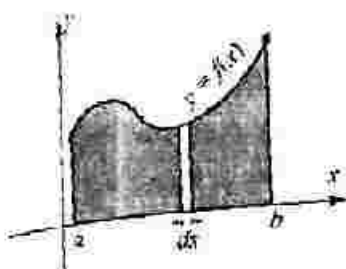
$$\Rightarrow F'(x) = \frac{2x}{x^2 + 1}$$

$\therefore$ Answer: C

Application of Integral Calculus

A. Area between a curve and axis

If $y = f(x)$ is non-negative and continuous on $[a, b]$ then the area of the region bounded by the graph of $y = f(x)$ and by the line $x = a$ and $x = b$ is given by: Area (A) =



$$\int_a^b f(x) dx.$$

Note: Area = (height)(width) $\longrightarrow$

$$dA = f(x)dx \rightarrow A = \int_a^b f(x) dx.$$

Illustrative Example

62. Find the area A of the region between the graph of f the x -axis from $x = a$ to $x = b$ when:

- $f(x) = x^3 - 9x$, $x = -2$ and $x = 2$
- $f(x) = \frac{1}{x}$, $x = -6$ and $x = -2$
- $f(x) = \ln x$, $x = e$ and $x = e^3$
- $f(x) = 18 - 2x^2$, $x = -4$ and $x = 2$
- $f(x) = x^3$, $x = -1$ and $x = 2$
- $f(x) = 2x^2 - 3x + 2$, $x = 0$ and $x = 2$
- $g(x) = \frac{x}{x^2 + 1}$, $x = 0$ and $x = 3$

Solution: a. $A = A_1 + A_2$;

$$A_1 = \int_{-2}^0 (x^3 - 9x) dx = \left[\frac{x^4}{4} - \frac{9x^2}{2} \right]_{-2}^0 = 14$$

But on $(0, 2)$, $f(x) < 0 \Rightarrow A_2 = \int_0^2 (-x^3 + 9x) dx$

$$= \left[-\frac{x^4}{4} + \frac{9x^2}{2} \right]_0^2 = \frac{-16}{4} + \frac{36}{2} = 14$$

$$\therefore A = A_1 + A_2 = 14 + 14 = 28 \text{ unit square}$$

b. Since $f(x) < 0$ when $x \in (-6, -2)$ so that $f(x) = -\frac{1}{x}$.

$$\begin{aligned}\therefore A &= \int_{-6}^{-2} -\frac{1}{x} dx = -\int_{-6}^{-2} \frac{1}{x} dx = -\ln|x| \Big|_{-6}^{-2} \\ &= -\ln|2| + \ln|-6| = \ln 6 - \ln 2 = \ln\left(\frac{6}{2}\right) = (\ln 3) \text{ unit sq.}\end{aligned}$$

$$\text{c. } A = \int_e^{e^3} (\ln x) dx = (x \ln x - x) \Big|_e^{e^3} = (3e^3 - e^3) - (e - e) = 2e^3$$

$$\text{d. Since } f(x) < 0 \text{ on } [-4, -3] \Rightarrow A = A_1 + A_2$$

$$\begin{aligned}\therefore A &= \int_{-4}^{-3} (-18 + 2x^2) dx + \int_{-3}^2 (18 - 2x^2) dx \\ &= \left(-18x + \frac{2x^3}{3} \right) \Big|_{-4}^{-3} + \left(18x - \frac{2x^3}{3} \right) \Big|_{-3}^2 = (F(-3) - F(-4)) + (G(2) - G(-3)) \\ &= [F(-3) - F(-4)] + [G(2) - G(-3)] = \frac{20}{3} + \frac{200}{3} = \frac{220}{3}\end{aligned}$$

$$\text{e. } f(x) < 0 \text{ on } [-1, 0] \Rightarrow A = A_1 + A_2$$

$$\therefore A = \int_{-1}^0 -x^3 dx + \int_0^2 x^3 dx = \left. \frac{-x^4}{4} \right|_{-1}^0 + \left. \frac{x^4}{4} \right|_0^2 = \frac{1}{4} + 4 = \frac{17}{4}$$

$$\text{f. Since } f(x) > 0 \text{ on interval } [0, 2].$$

$$\begin{aligned}\therefore A &= \int_0^2 f(x) dx = \int_0^2 (2x^2 - 3x + 2) dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x \right]_0^2 \\ &= \left(\frac{16}{3} - 6 + 4 \right) - (0 - 0 + 0) = \frac{10}{3}\end{aligned}$$

$$\text{g. } f(x) > 0 \text{ on } [0, 3] \Rightarrow A = \int_0^3 \frac{x}{x^2 + 1} dx$$

$$\Rightarrow A = \frac{1}{2} \int_0^3 \frac{2x}{x^2 + 1} dx = \frac{1}{2} \ln(x^2 + 1) \Big|_0^3 = \frac{1}{2} \ln(10)$$

61. Find the area bounded by the graph of:

a. $f(x) = |x^2 - 4|$, x -axis and the line $x = -3$ and $x = 4$.

b. $f(x) = |x + 3|$, x -axis and the line $x = -4$ and $x = 1$.

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Solution:

$$|x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } x \leq -2 \text{ or } x \geq 2 \\ -x^2 + 4 & \text{if } -2 < x < 2 \end{cases} \Rightarrow A = \int_{-3}^4 |x^2 - 4| dx$$

$$\therefore \text{Area, } (A) = A_1 + A_2 + A_3 =$$

$$\int_{-3}^{-2} (x^2 - 4) dx + \int_{-2}^2 (4 - x^2) dx + \int_2^4 (x^2 - 4) dx$$

$$= \left[\frac{x^3}{3} - 4x \right]_{-3}^{-2} + \left[4x - \frac{x^3}{3} \right]_{-2}^2 + \left[\frac{x^3}{3} - 4x \right]_2^4 = \frac{71}{3}$$

b. $|x+3| = x+3$ if $x \geq -3$ and $|x+3| = -x-3$ if $x < -3$

$$\therefore A = \int_{-4}^1 |x+3| dx = \int_{-4}^{-3} (-x-3) dx + \int_{-3}^1 (x+3) dx = 25$$

62. Find the area A of the entire region R bounded by:

a. $f(x) = x^3 - x$ and x-axis

b. $f(x) = x^3 - x^2 - 2x$ and x-axis

Solution: $f(x) = x^3 - x < 0$ on $(0, 1) \Rightarrow f(x) = -x^2 + x$ and $x^3 - x = x(x-1)(x+1) = 0 \Rightarrow x = 0, -1, x = 1$

$$\therefore A = \int_{-1}^0 (x^3 - x) dx + \int_0^1 (-x^3 + x) dx = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

b. $x^3 - x^2 - 2x = 0 \Rightarrow x(x^2 - x - 2) = x(x-2)(x+1) = 0 \Rightarrow x = 0, 2, -1$
 since $f(x) = x^3 - x^2 - 2x < 0$ on $(0, 2) \Rightarrow f(x) = -x^3 + x^2 + 2x$.

$$\therefore A = \int_{-1}^0 (x^3 - x^2 - 2x) dx + \int_0^2 (-x^3 + x^2 + 2x) dx$$

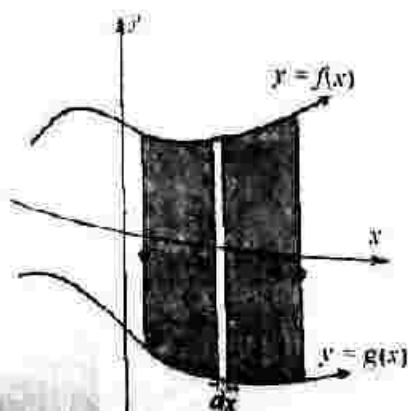
$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 + \left[-\frac{x^4}{4} + \frac{x^3}{3} + x^2 \right]_0^2 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

The Area between two curves

The area A bounded by two continuous curve $y = f(x)$ and $y = g(x)$ on $[a, b]$ with $f(x) \geq g(x)$, for all x in $[a, b]$ is given by:

$$A = \int_a^b (f(x) - g(x)) dx$$

- $f(x)$ is top curve
- $g(x)$ is bottom curve



$$\therefore A = \int_{-2}^2 [4 - x^2 - (3x^2 - 12)] dx = \int_{-2}^2 (-4x^2 + 16) dx$$

$$\Rightarrow A = \left. \frac{-4x^3}{3} + 16x \right|_{-2}^2 = \left(\frac{-32}{3} + 32 \right) - \left(\frac{32}{3} - 32 \right) = \frac{128}{3}$$

- e) **The intersection of g and f is $f(x) = g(x)$, then**
 $\Rightarrow x^2 + 2x + 1 = 2x + 5 \Leftrightarrow x^2 - 4 = 0 \Rightarrow x = -2$ and $x = 2$ since $g(x) \geq f(x)$ for all when $x \in [-2, 2]$.

$$\therefore A = \int_{-2}^2 [g(x) - f(x)] dx = \int_{-2}^2 [2x + 5 - (x^2 + 2x + 1)] dx$$

$$= \int_{-2}^2 (4 - x^2) dx$$

$$\therefore A = \left(4x - \frac{x^3}{3} \right) \Big|_{-2}^2 = \left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) = \frac{32}{3} \text{ unit sq.}$$

- d) **The intersection point of $f(x)$ and $g(x)$ is $f(x) = g(x)$**
 $\Rightarrow x^2 - 4x + 3 = -x^2 + 2x + 3 \Leftrightarrow 2x^2 - 6x = 0 \Rightarrow x = 0$; $x = 3$ since $g(x) \geq f(x)$ for all $x \in [0, 3]$.

$$\therefore A = \int_0^3 [g(x) - f(x)] dx = \int_0^3 (-x^2 + 2x + 3 - x^2 + 4x - 3) dx$$

$$\Rightarrow A = \int_0^3 (-2x^2 + 6x) dx = \left(\frac{-2x^3}{3} + 3x^2 \right) \Big|_0^3 = 9 \text{ unit sq.}$$

- e) **The intersection of $g(x)$ and $f(x)$ is $f(x) = g(x)$**
 $\Rightarrow 3x^2 - x^2 - 10x = -x^2 + 2x \Leftrightarrow 3x^3 - 12x = 0 \Rightarrow x = 0, x = -2, 2$
 since $f(x) \geq g(x)$ for all $x \in [-2, 0]$ but $g(x) \geq f(x)$ for all $x \in [0, 2]$

$$\Rightarrow A = \int_{-2}^0 [f(x) - g(x)] + \int_0^2 [g(x) - f(x)] dx$$

$$\Rightarrow A = \int_{-2}^0 [(3x^3 - x^2 - 10x) - (-x^2 + 2x)] dx$$

$$+ \int_0^2 (-x^2 + 2x - 3x^3 + x^2 + 10x) dx$$

$$\therefore A = \int_{-2}^0 (3x^3 - 12x) dx + \int_0^2 (-3x^3 + 12x) dx$$

$$= \left. \frac{3x^4}{4} - 6x^2 \right|_{-2}^0 + \left. \frac{-3x^4}{4} + 6x^2 \right|_0^2 = 12$$

Exercise left for you (Tip)

a) The intersection of $f(x)$ and $g(x)$ is $f(x) = g(x)$

$$\Rightarrow x^2 - 3x = 4 \Leftrightarrow x^2 - 3x - 4 = 0 \Rightarrow x = -1 \text{ and } x = 4 \text{ since}$$

$$f(x) \geq g(x) \text{ for all } x \in [-1, 4].$$

$$\begin{aligned} \Rightarrow A &= \int_{-1}^4 [g(x) - f(x)] dx = \int_{-1}^4 (4 - x^2 + 3x) dx \\ &= 4x - \frac{x^3}{3} + \frac{3x^2}{2} \Big|_{-1}^4 = \frac{143}{6} \end{aligned}$$

65. Find the area of the region bounded by:

a. $x = 8 - y^2$ and $x = y^2 - 8$ c. $y = \frac{1}{2}x$ and $x = 8$

b. $x = 3 - y^2$ and $y = x - 1$

Solution: Area, $A = \int_c^d x dy$ where $x = f(y)$. Here the roles of x and y is reverse to make easy so that to find the area, it is better to integrate with respect to y . Then use:

$$A = \int_c^d [f(y) - g(y)] dy \text{ for } f(y) \geq g(y), \text{ for all } y \in [c, d]$$

a) Let $f(y) = 8 - y^2$ and $g(y) = y^2 - 8$. The intersection of $f(y)$ and $g(y)$ is $f(y) = g(y) \Rightarrow 8 - y^2 = y^2 - 8 \Rightarrow 2y^2 = 16$

$$\Rightarrow y = -2\sqrt{2} \text{ and } y = 2\sqrt{2} \text{ since}$$

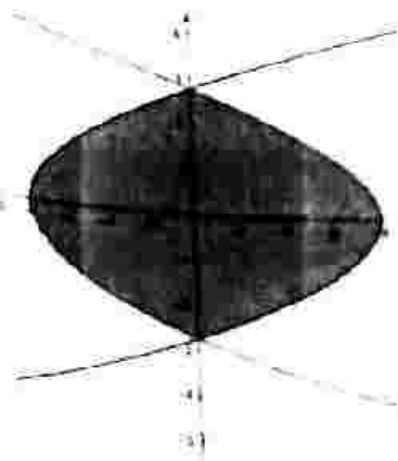
$$f(y) \geq g(y) \text{ for all}$$

$$y \in [-2\sqrt{2}, 2\sqrt{2}]$$

$$\Rightarrow A = \int_{-2\sqrt{2}}^{2\sqrt{2}} [f(y) - g(y)] dy$$

$$= \int_{-2\sqrt{2}}^{2\sqrt{2}} (8 - y^2 - y^2 + 8) dy = \int_{-2\sqrt{2}}^{2\sqrt{2}} (16 - 2y^2) dy$$

$$= \int_{-2\sqrt{2}}^{2\sqrt{2}} (16 - 2y^2) dy = \frac{128\sqrt{2}}{3}$$



- b) Let $f(y) = 3 - y^2$ and $y = x - 1 \Rightarrow x = y + 1 \Rightarrow g(y) = y + 1$. The intersection point is $f(y) = g(y) \Rightarrow 3 - y^2 = y + 1 \Leftrightarrow y^2 + y - 2 = 0 \Rightarrow y = -2$ and $y = 1$. Since $f(y) \geq g(y)$ for all $y \in [-2, 1]$

$$\Rightarrow A = \int_{-2}^1 (f(y) - g(y)) dy = \int_{-2}^1 (3 - y^2 - y - 1) dy$$

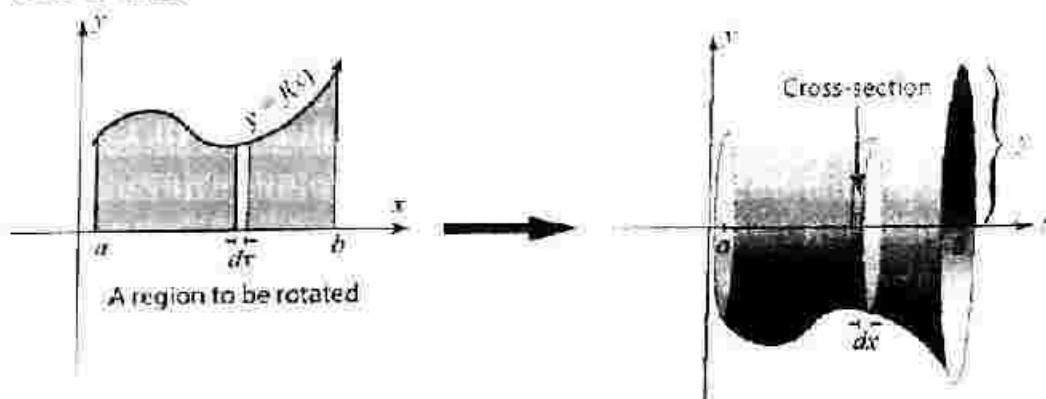
$$\therefore A = \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y \right]_{-2}^1 = \left[\left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) \right] = \frac{9}{2}$$

- c) Exercise left for you. (Answer 36 sq. unit)

II. Volume of Revolution

Another important application of integration is in finding the volume of three dimensional solid. If the region in the plane is revolved (rotated) about a line the resulting solid is called **solid revolution** and the line is called the **axis of revolution**. The volume of solid generated by rotating is called **volume of revolution**.

Methods of determining the volume of a solid of revolution about the x-axis



A solid revolution

To find the volume:

- In precalculus (volume of cylinder) = (base area)(height) = $\pi r^2 h$.
- In calculus the volume generated by rotating infinite small rectangle is:
 $\Delta v = (\text{Area of circular cross section})(\text{width of cross section})$
 $\Delta v = (\pi y^2) \Delta x \Rightarrow dv = \pi y^2 dx$, but
 $y = f(x) \Rightarrow dv = \pi (f(x)^2) dx \leftarrow \text{infinite small volume}$
- The volume of a solid can be obtained by integrating the cross sectional area from one end of the solid to other. The volume of a

solid revolution with disk (circular cross section) method is given

by:

$$\text{volume} = v = \int_a^b \pi (f(x))^2 dx \leftarrow \text{Horizontal axis of revolution (about x-axis).}$$

$$\text{volume} = v = \int_c^d \pi [f(y)]^2 dy \leftarrow \text{revolution about y-axis (vertical)}$$

Illustrative Example

66. Find the volume of the solid of revolution generated by revolving the region bounded by the given function about: i) the x axis

ii) the y axis

a. $y = 3x$; $x = 0$ and $x = 1$ c. $y = x^3$, $x = -2$ and $x = 1$

b. $y = -x + 1$, $x = -2$ and $x = 1$ d. $y = x^2 + 2$, $x = -1$ and $x = 2$

Solution: Here

a. i) volume about x-axis: $v = \int_a^b \pi [f(x)]^2 dx$

$$\Rightarrow v = \int_0^1 \pi (3x)^2 dx = \int_0^1 \pi 9x^2 dx = \pi [3x^3]_0^1 = 3\pi$$

ii) volume revol. about y axis: $v = \int_c^d \pi [f(y)]^2 dy$,

but $f(y) = x$ since $y = 3x \Rightarrow x = \frac{y}{3} \Rightarrow f(y) = \frac{y}{3}$ and

if $x = 0 \Rightarrow y = 0$ and if $x = 1$ then $y = 3$.

$$\therefore v = \int_0^3 \pi (f(y))^2 dy = \int_0^3 \pi \left(\frac{y}{3} \right)^2 dy = \pi \left[\frac{y^3}{27} \right]_0^3 = \pi$$

b. i) volume revol. about x axis: $y = -x + 1 \Rightarrow f(x) = -x + 1$

$$\Rightarrow v = \int_{-2}^1 \pi (-x + 1)^2 dx = \int_{-2}^1 \pi (x^2 - 2x + 1) dx$$

$$= \pi \left[\frac{x^3}{3} - x^2 + x \right]_{-2}^1 = 9\pi$$

ii) vol. revolution about y axis: since $y = -x + 1$

$$\Rightarrow x = 1 - y \Rightarrow f(y) = 1 - y. \text{ Then vol} = \int_c^d \pi [f(y)]^2 dy$$

when $x = 1 \Rightarrow y = 0$ and when $x = -2 \Rightarrow y = 3$.

$$\begin{aligned}\therefore v &= \int_0^3 \pi(1-y)^2 dy = \int_0^3 \pi(1-2y+y^2) dy \\ &= \pi \left[y - y^2 + \frac{y^3}{3} \right]_0^3 = 3\pi\end{aligned}$$

c.i) **vol. about x-axis** since $y = x^3 \Rightarrow f(x) = x^3$

$$\Rightarrow v = \int_{-2}^1 \pi(x^3)^2 dx = \pi \left[\frac{x^7}{7} \right]_{-2}^1 = \pi \left[\frac{1}{7} + \frac{128}{7} \right] = \frac{129\pi}{7}$$

ii) **vol. about y-axis** since $y = x^3 \Rightarrow x = \sqrt[3]{y}$

$\Rightarrow f(y) = \sqrt[3]{y}$ and when $x=1 \Rightarrow y=1$ and $x=-2 \Rightarrow y=-8$

$$\therefore v = \int_{-8}^1 \pi[f(y)]^2 dy = \int_{-8}^1 \pi(y^{\frac{1}{3}})^2 dy = \frac{3\pi}{5} (y^{\frac{5}{3}})_{-8}^1 = \frac{99\pi}{5}$$

d. *Exercise left for you (tip).*

67. Find the volume of the solid formed by revolving the region bounded by the given graph of function about x-axis.

a. $f(x) = \ln x$, $x=1$ and $x=e$

b. $f(x) = \sin x$, $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$

c. $f(x) = \cos x$, $x=0$, and $x = \frac{\pi}{3}$

d. $f(x) = 3^x$, $x=-2$ and $x=1$

e. $f(x) = e^x + e^{-x}$, $x=-1$ and $x=1$

Solution: a) $v = \int_1^e \pi[f(x)]^2 dx = \int_1^e \pi(\ln x)^2 dx$, let $u = (\ln x)^2$

$$\Rightarrow du = \frac{2 \ln x}{x} dx \text{ and } dv = dx \Rightarrow v = x.$$

$\therefore v = \int_1^e \pi(\ln x)^2 dx = \pi \left[x(\ln x)^2 - 2 \int \ln x dx \right] \leftarrow \text{using integration by parts.}$

$$\Rightarrow v = \pi \left[x(\ln x)^2 - 2(x \ln x - x) \right] \Big|_1^e = (e-2)\pi$$

$$b) V = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \pi (\sin x)^2 dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \pi \sin^2 x dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \pi \frac{(1 - \cos 2x)}{2} dx$$

Using $\cos 2x = \cos^2 x - \sin^2 x$ and $\cos^2 x + \sin^2 x = 1$.

$$\Rightarrow V = \left[\pi \left(\frac{1}{2} x - \frac{\sin 2x}{4} \right) \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi^2}{6} - \frac{\pi^2}{8} + \frac{\pi}{8} = \frac{\pi^2 + 3\pi}{24}$$

c) Exercise left for you (tip).

$$d) \text{ vol} = \int_{-2}^1 \pi (3^x)^2 dx = \int_{-2}^1 \pi (3^{2x}) dx = \pi \left[\frac{3^{2x}}{2 \ln(3)} \right]_{-2}^1$$

$$\therefore V = \frac{\pi}{2} \left[\frac{3^2 - 3^{-4}}{\ln 3} \right] = \frac{4\pi}{81 \ln 3}$$

$$e) \text{ vol} = \pi \int_{-1}^1 (e^x + e^{-x})^2 dx = \pi \int_{-1}^1 (e^{2x} + 2 + e^{-2x}) dx$$

$$\therefore V = \pi \left[\frac{e^{2x}}{2} + \frac{-e^{-2x}}{2} + 2x \right]_{-1}^1 = \pi \left[\frac{e^2}{2} - \frac{e^{-2}}{2} + 2 \right] - \left(\frac{e^2}{2} - \frac{e^{-2}}{2} - 2 \right)$$

$$\therefore V = \pi \left[\frac{e^{2x}}{2} - \frac{e^{-2x}}{2} + 2x \right]_{-1}^1 = \pi \left[\frac{e^2}{2} - \frac{e^{-2}}{2} + 2 - \left(\frac{e^2}{2} - \frac{e^{-2}}{2} - 2 \right) \right]$$

$$= (e^2 - e^{-2} + 4)\pi$$

Volume of sphere

- Verify that the volume of sphere is $\frac{4}{3}\pi r^3$ sphere with radius r is generated by rotating a semicircle; $x^2 + y^2 = r^2$

$$\Rightarrow y = \sqrt{r^2 - x^2}$$

$$\therefore V = \int_{-r}^r \pi (\sqrt{r^2 - x^2})^2 dx = \pi \int_{-r}^r (r^2 - x^2) dx$$

$$\Rightarrow V = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r = \pi (2) \left[r^2 x - \frac{x^3}{3} \right]_0^r = 2\pi \left(r^3 - \frac{r^3}{3} \right) = \frac{4\pi r^3}{3}$$

Illustrative Example

68. Find the volume of hemisphere generated by circle:

a. $x^2 + y^2 = 9$

b. $x^2 + y^2 = 1$

Solution: a) Since $r = 3$, $\Rightarrow v = \frac{2}{3} \pi r^3 = \frac{2}{3} \pi (9) = 6\pi$

b) Since $r = 1 \Rightarrow v = \frac{2}{3} \pi r^3 = \frac{2}{3} \pi$

69. A bowl shaped like hemisphere of radius 10 m contains some water. If the radius of surface of water is 6m, then find the volume of water.

Solution: A hemisphere generated by rotating the quarter of circle whose equation $x^2 + y^2 = 10^2$.

$$\Rightarrow x = \sqrt{100 - y^2}, 0 \leq x \leq 10, \text{ and } 0 \leq y \leq 10$$

- To find the limit of integration:
since radius of water, 6 m, so let $x = 6 \Rightarrow 36 + y^2 = 100$

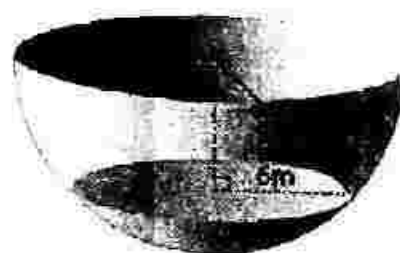
$$\Rightarrow y = \sqrt{100 - 36} = 8\text{m and when } x = 0 \Rightarrow y = 10.$$

$\therefore$ Volume of water,

$$v = \int_8^{10} \pi \left(\sqrt{100 - y^2} \right)^2 dy = \int_8^{10} \pi (100 - y^2) dy$$

$$\Rightarrow v = \pi \left[100y - \frac{y^3}{3} \right]_8^{10} = \pi \left[\left(1000 - \frac{1000}{3} \right) - \left(800 - \frac{512}{3} \right) \right]$$

$$= \frac{112\pi\text{m}^3}{3}$$



Hemispherical bowl

70. A bowl shaped like hemisphere of radius 8m contains some water to depth of the water is 3m, then find:

a. Volume of water

b. the radius of the surface of water

Solution: A hemisphere generated by rotating the quarter of the circle whose equation $x^2 + y^2 = 8^2 \Rightarrow x = \sqrt{64 - y^2}, 0 \leq y \leq 8$ and $0 \leq x \leq 8$.

To find the limit of integration:

$$\therefore \text{volume of water: } \int_5^8 \pi \left(\sqrt{64 - y^2} \right)^2 dy = \int_5^8 \pi (64 - y^2) dy$$

⇒ vol. of water

$$\pi \left[64y - \frac{y^3}{3} \right]_5^8 = \left(512 - \frac{512}{3} - \left(320 - \frac{125}{3} \right) \right) \pi = 63\pi \text{ m}^3$$

The radius of the surface of water is obtained when

$$y=5 \Rightarrow x = \sqrt{64-25} = \sqrt{39} \leftarrow \text{radius of water surface.}$$

71. Find the volume of frustum of a right circular cone has radii R and r , and height h in terms of h , R and r .

Solution: Let $(0, r)$ and (h, R) point on the line to be rotated. The

equation of the line to be rotated is $\frac{y-r}{x-0} = \frac{R-r}{h}$

$$\Rightarrow y = \frac{R-r}{h}x + r; \text{ Now the frustum generated by rotating the}$$

graph of $y = \left(\frac{R-r}{h} \right)x + r$ on $[0, h]$ about the x -axis

$$v = \int_0^h \pi \left(\frac{R-r}{h}x + r \right)^2 dx = \frac{\pi h}{3(R-r)} [R^3 - r^3]$$

$$\therefore v = \frac{\pi h}{3(R-r)} ((R-r)(R^2 + Rr + r^2)) = \frac{\pi h}{3} [R^2 + Rr + r^2]$$

72. Find the volume of the cone, from #71.

Solution: Set $\frac{y}{x} = \frac{R}{h} \Rightarrow y = \frac{R}{h}x, \Rightarrow v = \int_0^h \pi \left(\frac{R}{h}x \right)^2 dx$

$$\Rightarrow v = \pi \left[\frac{R^2 x^3}{h^2 \cdot 3} \right]_0^h = \frac{\pi R^2 (h^3)}{3h^2} = \frac{1}{3} \pi R^2 h$$

Volume of a solid or revolution generated by two curves (Washer cross sections)

If r and R are the inner and outer radii of the washer circular cross section and Δx is the width of cross-section then volume generated by two curves is given by:

Volume of cross section = $\pi(R^2 - r^2)\Delta x$

- Let the outer radius: $R = f(x)$

- Let the inner radius $r = g(x)$ then the volume of solid revolution is given by:

$$V = \int_a^b \pi [f(x)]^2 - [g(x)]^2 dx \text{ if } f(x) \geq g(x)$$

73. Find the volume of solid generated by revolving the region enclosed by the graphs of the equations about the x -axis.
- $f(x) = x^2 + 2$ and $g(x) = 3$
 - $f(x) = x^2$ and $g(x) = 4x - x^2$
 - $f(x) = \sqrt{x}$ and $g(x) = x^3$

Solution: First determine the intersection of f and g (set $f(x) = g(x)$)

a) $f(x) = g(x) \Leftrightarrow x^2 + 2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = -1$ and $x = 1$

$$\Rightarrow \text{vol} = \int_{-1}^1 \pi [(g(x))^2 - (f(x))^2] dx = \int_{-1}^1 \pi (3^2 - (x^2 + 2)^2) dx$$

$$\Rightarrow \text{vol} = \int_{-1}^1 \pi (5 - x^4 - 4x^2) dx = \pi \left[5x - \frac{x^5}{5} - \frac{4x^3}{3} \right]_{-1}^1 = \frac{104\pi}{15}$$

b) **Intersection:** $f(x) = g(x) \Rightarrow x^2 = 4x - x^2 \Rightarrow 4x - 2x^2 = 0 \Rightarrow 2x(2 - x) = 0$
 $\Rightarrow x = 0$ and $x = 2$, since $g(x) \geq f(x)$, for all $x \in [0, 2]$

$$\therefore V = \int_0^2 \pi [(4x - x^2)^2 - (x^2)^2] dx = \int_0^2 \pi (16x^2 - 8x^3) dx$$

$$V = \pi \left(\frac{16}{3} x^3 - 2x^4 \right) \Big|_0^2 = \pi \left(\frac{128}{3} - 32 \right) = \frac{32}{3} \pi$$

c. **The intersection of f and g :** $f(x) = g(x) \Leftrightarrow \sqrt{x} = x^3 \Rightarrow x = x^6$
 $\Rightarrow x^6 - x = 0 \Rightarrow x(x^5 - 1) = 0 \Rightarrow x = 0$ and $x = 1$ since $f(x) \geq g(x)$ on $[0, 1]$.

$$\therefore V = \int_0^1 \pi \left((\sqrt{x})^2 - (x^3)^2 \right) dx = \int_0^1 \pi (x - x^6) dx$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^7}{7} \right]_0^1 = \frac{5\pi}{14}$$

74. Find the volume of solid of revolution about the y -axis generated by the region enclosed by the graphs of the given function.

a. $y = x^2$ and $y = 2x$ rotates about y -axis.

b. $y = e^x$, y -axis and $y = e$, rotates about y -axis.

Solution: a) Intersection of $y = x^2$ and $y = 2x$ is $x^2 = 2x$

$$\Rightarrow x^2 - 2x = 0 \Leftrightarrow x(x - 2) = 0 \Rightarrow x = 0 \text{ and } x = 2$$

Since about y -axis, so that y -axis so that $y = x^2 \Rightarrow x = \sqrt{y}$ and y

$$= 2x \Rightarrow x = \frac{y}{2}, \text{ also limit of integration when } x = 0 \Rightarrow y = 0 \text{ and } x$$

$$= 2 \Rightarrow y = 4.$$

$$\therefore V = \int_0^4 \pi \left((\sqrt{y})^2 - \left(\frac{y}{2} \right)^2 \right) dy = \int_0^4 \pi \frac{1}{2} \left(y - \frac{y^2}{4} \right) dy$$

$$\therefore V = \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4 = \pi \left[8 - \frac{64}{12} \right] = \frac{8\pi}{3}$$

b) The intersection of $y = e^x$ and $y = e$ is $e^x = e \Rightarrow x = 1$ and when $x = 0 \Rightarrow y = 1$, since about rotation about y -axis, so that $y =$

$$e^x \Rightarrow x = \ln y. \therefore \text{vol} = \int_1^e \pi (\ln y)^2 dy = y(\ln y)^2 - 2 \int \ln y dy$$

$$= \pi (y(\ln y)^2 - 2y \ln y + 2y) \Big|_1^e = (e - 2)\pi \text{ cubic unit}$$

Work done by variable force

If a variable force is applied to an object calculus is needed to determine the work done because the amount of force changes as the object changes position, for example the force required to compress the spring.

Work done by a variable force as the object is moved from $x = a$ to $x = b$

$$\text{is } W = \int_a^b F(x) dx.$$

Motion of a particle in a straight line

$$\text{We know that the velocity } (v) = \frac{\Delta s}{\Delta t} \Rightarrow v(t) = s'(t) = \frac{ds}{dt}$$

$$\Rightarrow a(t) = v'(t) = \frac{dv}{dt} \Rightarrow a(t) dt = dv \Rightarrow v = \int a(t) dt$$

Example: A particle moves along straight line with velocity $v(t) = t^2 - 3t$, then find the displacement if particle has its position 4m when the time $t = 0$.

$$\text{Solution: } s(t) = \int v(t) dt = \int (t^2 - 3t) dt = \frac{t^3}{3} - \frac{3t^2}{2} + c \text{ and } s(0) = 4$$

$$\Rightarrow s(t) = \frac{t^3}{3} - \frac{3t^2}{2} + c \Rightarrow s(0) = 0 - 0 + c = 4 \Rightarrow c = 4$$

$$\therefore s(t) = \frac{t^3}{3} - \frac{3t^2}{2} + 4$$

Solved Problem

75. The volume of the solid obtained when the graph of $y = \sin x$, on $[0, \pi]$ is rotated about the x -axis is:

A. $\frac{1}{2}\pi^2$ B. $\frac{1}{2}\pi + 1$ C. $\frac{1}{2}\pi^2 - 1$ D. $\frac{1}{2}\pi$

Solution: $v = \int_0^\pi \pi \left(\frac{1}{2} - \frac{\cos 2x}{2} \right) dx = \pi \left[\frac{1}{2}x - \frac{\sin 2x}{4} \right]_0^\pi - \frac{1}{2}\pi^2$

∴ Answer: A

76. The volume of solid obtained by revolving the region bounded by the graph of $f(x) = x^2$ and $g(x) = 4$ about the x -axis ... (UEE)

A. $\frac{256\pi}{5}$ B. $\frac{512\pi}{15}$ C. $\frac{32\pi}{3}$ D. $\frac{16}{5}\pi$

Solution: The intersection of f and g is $f(x) = g(x) \Rightarrow x^2 = 4$

$\Leftrightarrow x = -2$ and $x = 2$. $v = \int_{-2}^2 \pi(4^2 - (x^2)^2) dx = \int_{-2}^2 \pi(16 - x^4) dx$

$$= \pi \left(16x - \frac{x^5}{5} \right) \Big|_{-2}^2 = \frac{256\pi}{5}, \therefore \text{Answer: A}$$

77. The area of the region enclosed by the graph of $y = x^3 - x$ and $y = 2x$.

A. $\frac{7}{2}$ B. $\frac{3}{4}$ C. $\frac{9}{2}$ D. $\frac{61}{105}$

Solution: The intersection of $y = x^3 - x$ and $y = 2x$ is solving $x^3 - x = 2x$

$\Rightarrow x^3 - 3x = 0 \Rightarrow x(x^2 - 3) = 0 \Rightarrow x = 0, x = -\sqrt{3}$ and

$x = \sqrt{3}$. Area of region (A) = $|A_1| + |A_2|$

$$\Rightarrow A = \int_{-\sqrt{3}}^0 (x^3 - x - 2x) dx + \int_0^{\sqrt{3}} (2x - x^3 + x) dx$$

$$\begin{aligned}
 &= \int_{-\sqrt{3}}^0 (x^3 - 3x) dx + \int_0^{\sqrt{3}} (-x^3 + 3x) dx \\
 &= \left(\frac{x^4}{4} - \frac{3x^2}{2} \right) \bigg|_{-\sqrt{3}}^0 + \left(-\frac{x^4}{4} + \frac{3x^2}{2} \right) \bigg|_0^{\sqrt{3}} \\
 &= \left(0 - \left(\frac{9}{4} - \frac{9}{2} \right) \right) + \left(\frac{9}{4} + \frac{9}{2} \right) = \frac{9}{2}
 \end{aligned}$$

78. The area of the region enclosed by the graph of $y = \sin x$ and $y = 0$ on $x \in [0, 2\pi]$.

A. 4 B. 2 C. 3 D. 0

Solution: The intersection will be $\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$.

$$\therefore A = \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} (0 - \sin x) dx = -\cos x \bigg|_0^{\pi} + \cos x \bigg|_{\pi}^{2\pi}$$

$$\Rightarrow A = -(-1 - 1) + (1 - (-1)) = 4$$

$\therefore$ Answer: A

79. The area of the region bounded by the graph of $f(x) = \sqrt{2-x}$ and $g(x) = -x$ and x -axis.

Solution: The intersection of $f(x)$ and $g(x)$ is $f(x) = g(x)$

$$\Rightarrow \sqrt{2-x} = -x \Leftrightarrow 2-x = x^2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow x^2 + x - 2 = 0 \Rightarrow x$$

$$-2, \text{ and } f(x) \text{ and } x\text{-axis intersects when } \sqrt{2-x} = 0 \Rightarrow x = 2$$

$$\therefore A = \int_{-2}^0 (\sqrt{2-x} + x) dx + \int_0^2 \sqrt{2-x} dx$$

$$= -\frac{2}{3} \sqrt{2-x}^3 \bigg|_{-2}^0 + \frac{x^2}{2} \bigg|_{-2}^0 = -\frac{2}{3} \sqrt{2-x}^3 \bigg|_0^2$$

$$= \frac{-2\sqrt{8}}{3} + \frac{16}{3} + -2 + \frac{2\sqrt{8}}{3} = \frac{10}{3}$$

80. The area of the region bounded between the graph of $y = x^2$ and $y = \sqrt{x}$ and is . . . UEE 2003

A. 3 sq. unit B. $\frac{1}{3}$ sq. unit C. $\frac{2}{3}$ sq. unit D. 1 sq. unit

Solution: Intersection of x^2 and $\sqrt{x}$ is $x^2 = \sqrt{x} \Rightarrow x = 0$ and $x = 1$.

$$\therefore A = \int_0^1 (\sqrt{x} - x^2) dx = \left(\frac{2}{3} \sqrt{x^3} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Answer: B

81. $\int_0^1 \frac{(x-1)^2}{x+1} dx$ equal to ... UEE 2004

A. $-\frac{3}{2} + \ln(6)$ B. $-\frac{5}{2} + \ln(15)$ C. $-\frac{5}{2} + 4 \ln 2$ D. $\frac{5}{2} + \ln 16$

Solution:

$$\begin{aligned} \int_0^1 \frac{(x-1)^2}{x+1} dx &= \int_0^1 \frac{x^2 - 2x + 1}{x+1} dx = \int_0^1 \left((x-3) + \frac{4}{x+1} \right) dx \\ &= \left(\frac{x^2}{2} - 3x + 4 \ln |x+1| \right) \Big|_0^1 = \frac{1}{2} - 3 \ln + 4 \ln 2 - 0 = -\frac{5}{2} + 4 \ln 2 \end{aligned}$$

Answer: C

82. The area of the region enclosed by the graph of $y = x$, $y = \frac{1}{x}$ and $y = 4$ is equal to:

A. $\frac{1}{2}(15 - \ln 4)$ C. $\frac{1}{2}(15 - \ln 8)$
B. $\frac{1}{2}(15 - \ln 16)$ C. $\frac{1}{2}(15 + \ln 8)$

Solution: The intersection $y = x$, and $y = \frac{1}{x}$ is $(1, 1)$, $(4, 4)$ and

$\left(\frac{1}{4}, 4\right)$, simply to integrate, use with respect to dy :

$$\begin{aligned} \therefore A &= \int_1^4 \left(y - \frac{1}{y} \right) dy = \left(\frac{y^2}{2} - \ln y \right) \Big|_1^4 = 8 - \ln 4 - \frac{1}{2} \\ &= \frac{1}{2}(15 - 2 \ln 4) = \frac{1}{2}(15 - \ln 16) \end{aligned}$$

Answer: B

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83. $\int_0^1 \frac{1}{(x+1)(x+2)} dx$ is equal to:

- A. $\log\left(\frac{2}{3}\right)$ B. $\log\left(\frac{4}{3}\right)$ C. 1 D. $\log\left(\frac{3}{4}\right)$

Solution: $\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \Leftrightarrow A(x+2) + B(x+1) =$

1. $A = 1, B = -1. \int_0^1 \frac{1}{(x+1)(x+2)} dx = \int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{x+2}$

$= \ln \left| \frac{x+1}{x+2} \right| \Big|_0^1 = \ln \left(\frac{4}{3} \right)$

$\therefore$ Answer: B

84. $\int_0^1 \frac{dx}{\sqrt{x+3} - \sqrt{x}}$ is equal to:

- A. $\frac{6-2\sqrt{3}}{3}$ B. $\frac{6+2\sqrt{3}}{3}$ C. $9-2\sqrt{3}$ D. $6-2\sqrt{3}$

Solution: Rationalizing the denominator, it gives

$\int_0^1 \frac{1(\sqrt{x+3} + \sqrt{x})}{(\sqrt{x+3} - \sqrt{x})(\sqrt{x+3} + \sqrt{x})} = \int_0^1 \frac{(\sqrt{x+3} + \sqrt{x})}{3} dx$

$= \frac{2}{9} \sqrt{x+3}^3 + \frac{2}{9} \sqrt{x}^3 \Big|_0^1 = 2 - \frac{2\sqrt{3}}{3} = \frac{6-2\sqrt{3}}{3} \therefore$ Answer: A

85. $\int_1^2 \log x dx$ is equal to:

- A. $2 \log 4 - 1$ B. $\log 4 + 1$ C. $\log 4 - 1$
D. $\log 4 + 4$

Solution: We know that $\log x = \frac{\ln x}{\ln 10}$ let $dv = dx$

$\Rightarrow v = \int dx = x$ and $u = \frac{\ln x}{\ln 10} \Rightarrow du = \frac{1}{x \ln 10}$

$\therefore \int_1^2 \log x dx = \int_1^2 \frac{\ln x}{\ln 10} = \left(\frac{x \ln x}{\ln 10} - \frac{x}{\ln 10} \right) \Big|_1^2 = \frac{2 \ln 2 - 1}{\ln 10}$

$$\Leftrightarrow \frac{2 \ln 2 - 1}{\ln 10} = \frac{\ln 4 - 1}{\ln 10} = \log 4 - 1 \quad \therefore \text{Answer: C}$$

Supplementary Exercise on Unit 14

1. Find

a. $\int \sqrt{3} dx$

b. $\int x^{\sqrt{2}} dx$

c. $\int \sqrt{3-4x} dx$

d. $\int \sqrt{2x-1} dx$

e. $\int (3x-1)^4 dx$

f.

$\int (3x+1)\sqrt[3]{3x+1} dx$

2. Find

a. $\int e^{4x} dx$

b. $\int e^{3x-4} dx$

c. $\int 3^{4x} dx$

d. $\int \frac{e^{2x} - e^{3x}}{e^{4x}} dx$

e. $\int \frac{5-e^x}{e^{2x}} dx$

f. $\int \frac{2^{i-2x}}{3^{x+1}} dx$

3. Find

a. $\int \frac{3}{x} dx$

b. $\int \frac{1}{4x-1} dx$

c. $\int \frac{3}{2-5x} dx$

d. $\int \frac{\sqrt{2}}{1-7x} dx$

e. $\int \frac{(e^x+1)^2}{e^x} dx$

f. $\int \left(\frac{1}{\sqrt{x+1}} - e^{2x} \right) dx$

g. $\int \left(\frac{1}{e^x} + \sqrt{e^x} \right) dx$

h. $\int \frac{1}{x^2} dx$

i. $\int \frac{1}{(x+3)^2} dx$

j. $\int \frac{1}{(x-4)^3} dx$

k. $\int \frac{1}{(2-3x)^2} dx$

4. Evaluate

a. $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}} dx$

b. $\int x^3 e^{x^4} dx$

c. $\int \frac{x+1}{\sqrt{x+3}} dx$

d. $\int \frac{6x}{\sqrt{x^2+4}} dx$

e. $\int (4x+1)(4x^2+2x+7)^6 dx$

f. $\int \frac{4x-1}{1-2x+4x^2} dx$

g. $\int (4x^2+12x+9)^4 dx$

h. $\int (16x^2-24x+9)^5 dx$

i. $\int 3x^2 \sqrt{x^2-2} dx$

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Evaluate the integral

a. $\int \frac{(\ln x)^2}{x} dx$ b. $\int \frac{1}{x(\ln x)^3} dx$ c. $\int \frac{1}{x \ln x} dx$

Evaluate the integral

a. $\int \frac{1}{4x^2 - 9} dx$ b. $\int \frac{x+2}{x^2 + 4x + 3} dx$ c. $\int \frac{5-x}{2x^2 + x - 1} dx$

d. $\int \frac{x^2 + 12x + 12}{x^3 - 4x} dx$ e. $\int \frac{x^2 - 1}{x^3 + x} dx$ f. $\int \frac{x}{16x^4 - 1} dx$

Evaluate the integral

a. $\int x^2 e^x dx$ b. $\int x^3 e^{-x} dx$ c. $\int x^2 e^{2x} dx$

d. $\int x \ln(x+1) dx$ e. $\int \ln 3x dx$ f. $\int (\ln x)^2 dx$

g. $\int x^2 \cos x dx$ h. $\int x^3 \cos 2x dx$ i. $\int x^3 \sin x dx$

Evaluate the integral

a. $\int e^x \sin 2x dx$ b. $\int e^{2x} \cos x dx$ c. $\int e^{-x} \sin x dx$

Find

a. $\int_0^3 |2x-3| dx$ b. $\int_1^4 (3 - |x-3|) dx$ c. $\int_0^3 |x^2 - 4| dx$

d. $\int_{-1}^1 x(x^2 + 1)^3 dx$ e. $\int_1^2 2x^2 \sqrt{x^3 + 1} dx$ f. $\int_0^4 \frac{1}{\sqrt{2x+1}} dx$

g. $\int_1^2 (x-1)\sqrt{2-x} dx$ h. $\int_0^{\frac{\pi}{2}} \cos\left(\frac{2x}{3}\right) dx$ i. $\int_0^1 \frac{1}{(2x+1)^{\frac{3}{2}}} dx$

Find

a. $\int_0^1 \frac{3}{2x^2 + 5x + 2} dx$ b. $\int_1^5 \frac{x-1}{x^2(x+1)} dx$ c. $\int_2^3 \frac{1}{x^2 - 1} dx$

d. $\int_0^1 x^2 e^x dx$ e. $\int_0^{\pi} x \sin 2x dx$ f. $\int_0^{\frac{\pi}{2}} x \cos x dx$

g. $\int_0^1 e^x \sin x dx$ h. $\int_0^2 e^{-x} \cos x dx$ i. $\int_1^2 x^2 \ln x dx$

Find

- a. $\int_{-\pi}^{\pi} \sin^2 x dx$ b. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x} dx$ c. $\int_0^{\pi} \cos^2 x dx$
 d. $\int_0^1 \frac{1-x}{1+x} dx$ e. $\int_0^1 \frac{1}{\sqrt{3-2x}} dx$ f. $\int_0^2 \frac{x^2}{\sqrt{x^3+1}} dx$
 g. $\int_0^1 \frac{1}{x+\sqrt{x}} dx$ h. $\int_0^2 x^2 e^x dx$ i. $\int_1^2 x \ln x dx$
 j. $\int_1^2 x^2 \sqrt{x^3+8} dx$ k. $\int_0^1 x^3 \sqrt{1+3x^4} dx$ l. $\int_0^{\frac{\pi}{2}} \cos^3 x dx$

12. Find the area of the region R bounded by the graph of each of the following (in square unit).

- a. $f(x) = 6 - x^2$ and $g(x) = x$, $\frac{125}{6}$
 b. $f(x) = \sin x$ and $g(x) = \cos x$, on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
 c. $f(x) = -x^2 + 2x$ and $g(x) = 3x^3 - x^2 - 10x$
 d. $f(x) = x^3 - 4x$ and $g(x) = 0$
 e. $f(x) = x^2 + 4x + 1$ and $g(x) = x + 1$
 f. $f(x) = 3^{2x}$, $x = 0$, $x = 1$ and $y = 0$

13. Find the area of the region bounded by:

- a. $x = y^2 - 6$ and $y = x$
 b. $y = \sqrt{x+2}$ and $y = x$
 c. $x = y^2 + 2y + 1$ and $x = 2y + 5$

14. Find the volume generated by the following graph about the x-axis (in cubic unit).

- a. $f(x) = -x + 2$, $x = 0$ and $y = 0$
 b. $f(x) = 4 - x^2$, $y = 0$ and $x = 0$
 c. $f(x) = 5x - x^2$, x-axis
 d. $f(x) = x^3$ and $g(x) = 4x$
 e. $f(x) = |x|$ and $g(x) = x^2$, from $x = -3$ to $x = 3$

15. Find the volume generated by the rotating the graph about the y-axis.

- a. $y = x^2 + 2$ and the line $y = 6$, rotates about y-axis.
 b. $y = x^2 - 1$ and the line $y = 2$, rotates about y-axis.
 c. $y = e^x$ the y-axis and the line $y = e$ rotates about y-axis.

Unit Fifteen

Three Dimensional Geometry

Here, we will discuss coordinate systems in three dimensional space and some basic facts of 3D.

Coordinate axes and coordinate planes in space

- The three mutually perpendicular lines, x -axis, y -axis and z -axis are called *three coordinate axis*.
- Taken as a pair, the axes determine three coordinate planes: The xy -plane, xz -plane and the yz -plane. These three coordinate planes intersect at origin. These three coordinate planes separate three-space into eight parts called **octants**. The set of points with three positive coordinate forms the first octant the remaining octants have no standard numbering.

Coordinates of a point in space

Recall a point P in plane:

- The point $P(-3, 4)$ means that p is 3 unit to the left of y -axis and 4 units above the x -axis.

In three dimensional system, a point $P(x, y, z)$ indicates:

- x = directed distance from yz plane to P
- y = directed distance from xz plane to P
- z = directed distance from xy plane to P .

The standard equation determined by the coordinate axes

It is possible to describe some geometric figures in space by means of equation. In xy -plane $x = 0$, in x -axis, $z = 0$, $y = 0$ in y -axis, $x = 0$, $z = 0$ and z -axis, $x = y = 0$.

Illustrative Example

1. Give the equation of the:

- | | | |
|----------------|----------------|----------------|
| a. xy -plane | b. xz -plane | c. yz -plane |
| d. x -axis | e. y -axis | f. z -axis |

Solution:

- xy -plane consists of all points of the form $(x, y, 0)$
- xz -plane consists of all points of the form $(x, 0, z)$

- c. yz -plane consists of all points of the form $(0, y, z)$
- d. x -axis consists of all points of the form $(x, 0, 0)$
- e. y -axis consists of all points of the form $(0, y, 0)$
- f. z -axis consists of all points of the form $(0, 0, z)$

Distance between two points in space

The distance between the point $P_1 (x_1, y_1, z_1)$ and $P_2 (x_2, y_2, z_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Illustrative Example

2. Find the distance between the points:

- a. $A (1, -2, 1)$ and $B (6, -2, -2)$ b. $A (2, -1, 3)$ and $B (1, 0, -2)$

Solution: Let d = distance b/n A and B , then:

a. $d = \sqrt{(6-1)^2 + (-2+2)^2 + (-2-1)^2} = \sqrt{25+0+9} = \sqrt{34}$

b. $d = \sqrt{(1-2)^2 + (0-(-1))^2 + (-2-3)^2} = \sqrt{1+1+25}$
 $= \sqrt{27} = 3\sqrt{3}$

3. Find the distance from the point $P (-3, 1, -2)$ to the

- a. $x = 1$ b. xy -plane c. xz -plane d. yz -plane
 e. z -axis f. y -axis g. x -axis h. $z = 0$

Solution: We know that, xy -plane is $(x, y, 0)$ or $z = 0$.

- a. $x = 1$, means the yz -plane meeting the x -axis at the point

$(1, 0, 0)$ is $d = \sqrt{(-3-1)^2 + (1-0)^2 + (-2-0)^2} = \sqrt{21}$

- b. xy -plane contain $(-3, 1, 0)$ set $z = 0$

$\Rightarrow d = \sqrt{(-3-(-3))^2 + (1-1)^2 + (-2-0)^2} = \sqrt{4} = 2$

- c. xz -plane contains $(-3, 0, -2)$ set $y = 0$

$\Rightarrow d = \sqrt{(-3-(-3))^2 + (1-0)^2 + (-2-(-2))^2}$
 $= \sqrt{0+1+0} = \sqrt{1} = 1$

- d. *Exercise left for you.* (Hint, set $x = 0$).

- e. z -axis contains $(0, 0, -2)$, thus distance between

$d = \sqrt{(3-0)^2 + (1-0)^2 + (-2-(-2))^2} = \sqrt{9+1} = 10$

- f. y -axis contains $(0, 1, 0) \leftarrow$ *Exercise left for you.*

- g. x -axis contains $(-3, 0, 0)$.

$$d = \sqrt{(-3 - (-3))^2 + (1 - 0)^2 + (-2 - 0)^2} = \sqrt{0 + 1 + 4} = \sqrt{5}$$

h. $z = 0$ is point on xy plane $\Rightarrow (-3, 1, 0)$

$$\text{so that } d = \sqrt{0 + 1 + 4} = \sqrt{5}$$

4. Find the possible coordinate of points on the x -axis which are $\sqrt{6}$ units from the point $(3, -1, 2)$.

Solution: x -axis contains $(x, 0, 0)$. The distance b/w $(x, 0, 0)$

$$\text{and } (3, -1, 2) \text{ is } \sqrt{6} \Rightarrow d = \sqrt{(x-3)^2 + (-1)^2 + (0-2)^2} = \sqrt{6}$$

$$\Rightarrow (x-3)^2 + 1 + 4 = 6 \Leftrightarrow (x-3)^2 = 1 \Rightarrow x-3 = 1 \text{ or } x-3 = -1$$

give $x = 4$ and $x = 2$.

$\therefore$ The coordinate points are $(4, 0, 0)$ and $(2, 0, 0)$.

5. Identify the type of triangle, isosceles, equilateral, right triangle if the vertices are given:

a. $A(4, 5, 2)$, $B(1, 7, 3)$ and $C(2, 4, 5)$

b. $A(0, 7, 10)$, $B(-1, 6, 6)$ and $C(-4, 9, 6)$

Solution: Here

$$a. \overline{AB} = \sqrt{(4-1)^2 + (5-7)^2 + (2-3)^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$\overline{BC} = \sqrt{(2-1)^2 + (4-7)^2 + (5-3)^2} = \sqrt{1+9+4} = \sqrt{14}$$

$$\overline{CA} = \sqrt{(4-2)^2 + (5-4)^2 + (2-5)^2} = \sqrt{4+1+9} = \sqrt{14}$$

$\therefore \triangle ABC$ is equilateral triangle.

b. Exercise left for you.

6. Find the set of all points on xz -plane which is equidistant from the point $(1, -1, 5)$ and $(5, 1, -1)$.

Solution: The set of all point on xz -plane is $(x, 0, z)$. Hence $(x, 0, z)$ is equidistant from the point $(1, -1, 5)$ and $(5, 1, -1)$.

$$\text{Thus } (x-1)^2 + (0+1)^2 + (z-5)^2 = (x-5)^2 + (0-1)^2 + (-1-z)^2$$

$\Rightarrow x^2 - 2x + 1 + 1 + z^2 - 10z + 25 = x^2 - 10x + 25 + 1 + z^2 + 2z + 1$ cancelling and simplifying similar terms, we get

$$8x - 12z = 0 \Rightarrow z = \frac{8}{12}x = \frac{2}{3}x$$

$$\therefore (x, y, z) = \{(x, 0, z)\} = \left\{ \left(x, 0, \frac{2}{3}x \right) \right\}, \text{ let } x = 3t.$$

$\therefore$ the set of all point on xz plane is $(3t, 0, 2t), t \in \mathbb{R}$

Equation of sphere

A sphere with center at (x_0, y_0, z_0) and radius r is defined to be the set of all point (x, y, z) such that the distance between (x, y, z) and (x_0, y_0, z_0) is the radius of sphere (r).

If (x, y, z) is an arbitrary point on the sphere the equation of sphere is $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

Illustrative Example

7. Find the equation of the sphere:

- a. Center: $(0, 2, 5)$, radius, $r = 2$
- b. Center: $(4, -1, 1)$, radius, $r = 5$
- c. End points of a diameter: $(2, 0, 0)$ and $(0, 6, 0)$

Solution: Let (x, y, z) be point on sphere, then:

a. $(x - 0)^2 + (y - 2)^2 + (z - 5)^2 = 4$

$$\Rightarrow x^2 + y^2 + z^2 - 4y - 10z + 25 = 0$$

b. $(x - 4)^2 + (y + 1)^2 + (z - 1)^2 = 25$

$$\Rightarrow x^2 + y^2 + z^2 - 8x + 2y - 2z - 7 = 0$$

c. Diameter

$$(d) = \sqrt{(0 - 2)^2 + (6 - 0)^2 + (0 - 0)^2} = \sqrt{40} = 2\sqrt{10}$$

$$\Rightarrow \text{radius, } r = \sqrt{10}$$

$$\text{center (midpoint)} = \left(\frac{2+0}{2}, \frac{0+6}{2}, \frac{0+0}{2} \right) = (1, 3, 0)$$

$$\therefore (x - 1)^2 + (y - 3)^2 + (z - 0)^2 = 10$$

$$\Rightarrow x^2 + y^2 + z^2 - 2x - 6y = 0 \text{ is equation of sphere.}$$

8. Find the equation of the sphere:

- a. Center: $(2, -1, -3)$, tangent to the xy plane
- b. Center: $(2, -1, -3)$ tangent to the xy -plane
- c. Center: $(-3, 2, 4)$ tangent to the yz plane

Solution: a) The distance b/n $(2, -1, -3)$ and xy -plane is 3

$$\Rightarrow (x - 2)^2 + (y + 1)^2 + (z + 3)^2 = 3^2$$

$$\Rightarrow x^2 + y^2 + z^2 - 4x + 2y + 6z + 5 = 0$$

b) $r = 1 \Rightarrow (x - 2)^2 + (y + 1)^2 + (z + 3)^2 = 1$

c) Exercise left for you.

9. Find the equation of sphere

- a. Center: $(1, 0, -1)$ and passing through $(2, 1, 3)$

- b. Center: $(-1, 3, 2)$ and passing through $(0, 0, 0)$

Solution: as radius

$$r = \sqrt{(2-1)^2 + (1-0)^2 + (3-(-1))^2} = \sqrt{18}$$

$$\Rightarrow (x-1)^2 + (y-0)^2 + (z+1)^2 = 18$$

b) Exercise left for you

Determining the distance between the point $P(x, y, z)$ and the sphere

10. How far is the point $P(x, y, z)$ from the sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.
How far is the sphere $(x-2)^2 + (y+1)^2 + (z-1)^2 = 4$ from the point P and which point is closer to sphere.

- a. $P(4, 0, 3)$ b. $P(-2, 2, 1)$ c. $P(0, 1, 3)$

Solution: Center of sphere $(2, -1, 1)$, with radius $r = 2$.

- a. The distance b/n the center $(2, -1, 1)$ and point $P(4, 0, 3)$.

$$\Rightarrow d = \sqrt{(4-2)^2 + (0+1)^2 + (3-1)^2} = \sqrt{4+1+4} = 3$$

$\therefore$ The distance b/n the sphere and point P is $3 - 2 = 1$

- b. The distance b/n the center $(2, -1, 1)$ and point $P(-2, 2, 1)$ is

$$d = \sqrt{(2-(-2))^2 + (-1-2)^2 + (1-1)^2} = \sqrt{16+9+0} = 5$$

$\therefore$ The distance b/n the sphere and point P is $5 - 2 = 3$.

- c. Distance b/n $(2, -1, 1)$ and $P(0, 1, 3)$ is

$$d = \sqrt{(3-2)^2 + (1+1)^2 + (3-1)^2} = 2\sqrt{3}$$

$\therefore$ The distance b/n the sphere and point is $2(\sqrt{3}-1)$. Note:

Therefore, $P(4, 0, 3)$ is closer to sphere than other point.

11. If the center of a sphere is at $(1, -2, 3)$ and radius $r = 3$ then determine which points lie inside or outside or on the sphere.

- a. $P(2, -1, 6)$ b. $P(3, 0, 4)$ c. $P(3, -1, 2)$

Solution: Equation of sphere with center $(1, -2, 3)$ and radius, $r = 3$ is $(x-1)^2 + (y+2)^2 + (z-3)^2 = 9$

- a. $(2, -1, 6)$: $(2-1)^2 + (-1+2)^2 + (6-3)^2 \Rightarrow 11 > 9$

$\therefore (2, -1, 6)$ is outside the sphere.

- b. $P(3, 0, 4)$: $(3-1)^2 + (0+2)^2 + (4-3)^2 \Rightarrow 9 = 9$

$\therefore P(3, 0, 4)$ is on the sphere.

- c. $P(3, -1, 2)$: $(3-1)^2 + (-1+2)^2 + (2-3)^2 \Rightarrow 6 < 9$

$\therefore P(3, -1, 2)$ is inside the sphere.

12. Which of the following is a point of intersection of the sphere

$$(x-1)^2 + (y-2)^2 + (z+1)^2 = 6 \text{ and the } y\text{-axis} \dots \text{UFE 2002}$$

- A. $(0, -2, 0)$ B. $(0, 2, 0)$ C. $(0, -4, 0)$ D. $(0, 4, 0)$

Solution: The equation on y -axis is $(0, y, 0)$

$$\Leftrightarrow (0-1)^2 + (y-2)^2 + (0+1)^2 = 6 \Rightarrow (y-2)^2 = 4$$

$$\Leftrightarrow (0-1)^2 + (y-2)^2 = 4 \Leftrightarrow y-2 = 2 \text{ and } y-2 = -2 \Rightarrow y = 4 \text{ and } y = 0. \therefore (0, 4, 0)$$

∴ Answer: D

13. Find the intersection of the sphere $(x-1)^2 + (y+2)^2 + (z-3)^2 = 8$ with: a. x -axis b. z -axis c. y -axis

Solution: a. x -axis $(x, 0, 0) \Rightarrow (x-1)^2 + (0+2)^2 + (0-3)^2 = 8$
 $\Rightarrow (x-1)^2 = 8 - 13 = -5$.

∴ There is no point of intersection with x -axis.

b. z -axis: $(0, 0, z) \Rightarrow (0-1)^2 + (0+2)^2 + (z-3)^2 = 8 \Leftrightarrow (z-3)^2 = 8 - 5 = 3 \Rightarrow z-3 = \sqrt{3} \text{ or } z-3 = -\sqrt{3}$.

∴ The sphere intersects at two point on $z = 3 + \sqrt{3}$ and $z = 3 - \sqrt{3}$. That is $(0, 0, 3 + \sqrt{3})$ and $(0, 0, 3 - \sqrt{3})$.

c. Exercise left for you.

14. Find the point of intersection of the sphere of the radius 3 unit with center at $(-1, 3, 2)$ and the z -axis. ... UEE 2003/11

Solution: $(x+1)^2 + (y-3)^2 + (z-2)^2 = 9$ and z -axis $(0, 0, z)$
 $\Rightarrow (0+1)^2 + (0-3)^2 + (z-2)^2 = 9 \Rightarrow (z-2)^2 = 9 - 10 = -1$
 $\Rightarrow (z-2)^2 = -1$. ∴ There is no point of intersection.

15. If a sphere with center $C(0, 1, 1)$ intersects the z -axis at $P(0, 0, 3)$ then the radius of the sphere is equal to ... UEE 2004/12

A. 5 B. 3 C. $\sqrt{3}$ D. $\sqrt{5}$

Solution: Eqn of sphere $(x-0)^2 + (y-1)^2 + (z-1)^2 = r^2$

$\Rightarrow (0-0)^2 + (0-1)^2 + (3-1)^2 = r^2 \Leftrightarrow r^2 = 5 \Rightarrow r = \sqrt{5}$. ∴ Ans. D

16. Suppose A and B are the end-points of a diameter of the sphere whose equation is $x^2 + y^2 + (z+2)^2 = 1$. If A $(1, 0, -2)$ then B is equal to ... UEE 2004/12

A. $(0, 1, -2)$ B. $(0, 0, -1)$ C. $(-1, 0, -2)$ D. $(0, -1, -2)$

Solution: Center of sphere $(0, 0, -2)$ as the mid-point of AB let the end-point of B (x, y, z) and A $= (1, 0, -2)$

$\left(\frac{1+x}{2}, \frac{0+y}{2}, \frac{-2+z}{2} \right) = (0, 0, -2) \leftarrow \text{midpoint of AB}$

$$\Rightarrow \frac{1+x}{2} = 0 \Rightarrow x = -1 \quad \text{and} \quad \frac{y+0}{2} = 0 \Rightarrow y = 0 \quad \text{and}$$

$$\frac{z-2}{2} = -2 \Rightarrow z = -2. \therefore B = (-1, 0, -2). \therefore \text{Answer: C}$$

17. If the line $z = 4 - \sqrt{3}y$ for $x = -1$ is tangent to the sphere whose equation is: $(x+1)^2 + y^2 + z^2 = 4$ then what is the distance of the line from the center of the sphere.

A. 2 B. 3 C. $2\sqrt{3}$ D. 4

Solution: $(-1+1)^2 + y^2 + (4 - \sqrt{3}y)^2 = 4$

$$\Rightarrow y^2 + 16 - 8\sqrt{3}y + 3y^2 = 4 \Rightarrow 4y^2 - 8\sqrt{3}y + 12 = 0$$

$$\Rightarrow y^2 - 2\sqrt{3}y + 3 = 0 \Rightarrow (y - \sqrt{3})^2 = 0 \Rightarrow y = \sqrt{3}, \text{ then}$$

$$z = 4 - \sqrt{3} \cdot \sqrt{3} = 4 - 3 = 1$$

$\therefore$ The distance b/n $(-1, \sqrt{3}, 1)$ and center $(-1, 0, 0)$ is

$$d = \sqrt{(-1+1)^2 + (\sqrt{3}-0)^2 + (1-0)^2} = \sqrt{4} = 2 \quad \therefore \text{Answer: A}$$

Supplementary Exercise on Unit - 15

- Find the distance from the point P (2, -1, 3) to the:
 - xy-plane
 - yz-plane
 - y-axis
 - z-axis
 - x-axis
 - x=0
- Find the possible coordinate of points on the y-axis which are $\sqrt{29}$ units from the point (4, -2, 3).
- Find the equation of sphere:
 - Center: (1, 0, -2) and passing through (3, 1, 2)
 - Center: (3, -1, 2) and passing through (0, 0, 4)
- How far is the point P (3, 0, 1) from the sphere $(x+1)^2 + y^2 + (z-4)^2 = 9$.

5. If the center of a sphere is at $(1, 0, 2)$ and radius $r = 3$, then determine which points lie inside, or outside or on the sphere.
 a. $(-2, 1, 3)$ b. $(3, -1, 4)$ c. $(2, -2, 3)$
6. Find the point of intersection of the sphere $(x+1)^2 + (y-3)^2 + z^2 = 4$ with:
 a. x -axis b. y -axis c. z -axis
7. Find the point of intersection of the sphere of the radius $\sqrt{38}$ unit with center at $C(2, 1, 3)$ with: a. y -axis b. x -axis
8. If a sphere with center $C(-3, 0, 3)$ intersects the z -axis at $(1, 0, 0)$ then the radius of the sphere is equal to _____.
9. For what value of k is the sphere:
 a. $x^2 + y^2 - 4x + kz = -4$ of radius 3 unit
 b. $x^2 + y^2 + z^2 - 2x + 4y + 6z + k = 0$ of radius $\sqrt{5}$ unit



Answer to Review Exercise on Unit - 1

a) Not function b) function c) Not function, since $a \geq 0$

d) Not function, since $A = \{1, 2, 3, 4\}$.

a) odd b) odd c) odd d) even

a) one-to-one correspondence

b) on to

c) one-to-one but no on to

d) one-to-one correspondence

e) one-to-one correspondence

f) one-to-one correspondence

g) on to but not one-to-one

h) on to

a) one-to-one c) one-to-one correspondence b)

one-to-one

a) $\left\{x : x \geq \frac{1}{2}\right\}$ b) $\{x : x \geq 2\}$ c) $\{x : x > 2\}$

a) True b) False c) False d) True

a) $f^{-1}(x) = e^{2(x-3)}$ b) $f^{-1}(x) = \frac{3(e^x - 1)}{e^x - 2}$

c) $f^{-1}(x) = 3 - \sqrt{\frac{x-1}{2}}$ d) $f^{-1}(x) = \ln x - \ln(1-x)$

$f(g(x)) = 2(3x+k)^2 + 5 = 45$, when $x=0 \Rightarrow 2(k^2) = 40$

$\Rightarrow k = 2\sqrt{5}$ or $k = -2\sqrt{5}$

$f(x+1) = 2x^2 - 7x + 4$

a) $\{x \in \mathbb{R} : 0 \leq x \leq 4\}$,

a) $f^{-1}(x) = -5 + \sqrt{x}$

c) $f^{-1}(x) = \frac{4x+1}{3x-1}$

e) $f^{-1}(x) = x^2 - 2$ for $x \geq 0$

b) $\{x \in \mathbb{R} : 0 \leq x \leq 4\}$

b) $f^{-1}(x) = \sqrt{x-3}$

d) $f^{-1}(x) = x^3 + 4$

f) $f^{-1}(x) = \sqrt[3]{x+2}$

Solution for Unit Three

1. a) $\sqrt{38-12\sqrt{6}}$
2. (12, 0) and (-4, 0)

b) $2\sqrt{a^2 + b^2}$

c) $|x| \sqrt{10}$

3. $y = -3 \pm \sqrt{119}$ 4. $x - y = 3$ 5. $x = 2$ 6. 4:1 (internally)
7. a) $\tan^{-1}\left(\frac{1}{8}\right)$ b) $\tan^{-1}\left(\frac{ab' - a'b}{aa' + bb'}\right)$ c) $\tan^{-1}\left(\frac{7}{4}\right)$
8. a) (4, 0) b) $x^2 + y^2 - 8x = 34$
9. (2, -4) 10. (9, 2), (4, 10), (-7, -6) 11. $k = 5$
12. $y + 2 = 0$, $\sqrt{3}x - y - (3\sqrt{3} + 2) = 0$ 13. $k = 0$, center = (-4, 3) and radius 4.
14. a) $(x - 2)^2 + y^2 = 9$ b) $x^2 + (y + 2)^2 = 25$
15. (a) $x^2 + y^2 + 2y - 12 = 0$ 16. a outside, B and C inside 17. $2\sqrt{5}$
18. Externally; (1, 0) 19. a) $5x - 4y = 11$, b) $10x + 3y + 20 = 0$
20. [Take center as (2a, a), radius = r] $x^2 + y^2 - 8x - 4y + 7 = 0$
21. $2y^2 - 9x = 0$ 22. a) latus rectum $4p = 6$

focus $\left(\frac{3}{2}, 0\right)$, vertex (0, 0), directrix $x = -\frac{3}{2}$

- b) latus rectum $|4p| = 9$ focus $\left(0, -\frac{9}{4}\right)$, vertex (0, 0) and directrix, $y = \frac{9}{4}$.
- c) latus rectum $4P = 6$, focus $\left(\frac{7}{2}, 3\right)$, vertex (2, 3), directrix, $x = \frac{1}{2}$
- d) latus rectum, $|4P| = 1$, focus $\left(-2, -\frac{9}{4}\right)$, vertex (-2, -2), directrix, $y = -\frac{7}{4}$
- e) latus rectum $|4P| = 2$, focus (2, 2), vertex $\left(2, \frac{5}{2}\right)$, directrix, $y = 3$
- f) latus rectum $|4P| = 1$, focus $\left(-\frac{7}{4}, 2\right)$, vertex (-2, 2), directrix $x = -\frac{9}{4}$
23. a) $y^2 = 12x$ c) $x^2 = -12y$ e) $(x + 1)^2 = 4(y - 2)$
- b) $y^2 = -28x$ d) $(x - 1)^2 = 12(y - 1)$ f) $(x + 4)^2 = 4(y - 1)$
- g) $(x - 3)^2 = 4(u + 4)$ 24. a) vertices (0, 5), (0, -5), foci (0, 3), (0, -3)
- b) vertices (3, 0), (-3, 0), foci $(\sqrt{5}, 0)$, $(-\sqrt{5}, 0)$
- c) vertices (0, 3), (0, -3), foci $(0, \sqrt{8})$, $(0, -\sqrt{8})$
- d) vertices (8, -2), (-2, -2), foci (6, -2), (0, -2)
- e) vertices (-2, 5), (-2, -5), foci (-2, 4), (-2, -4)
- f) vertices: $(1 + \sqrt{21}, 3)$, $(1 - \sqrt{21}, 3)$, foci $\left(0, \frac{\sqrt{55}}{2}\right)$, $\left(0, -\frac{\sqrt{55}}{2}\right)$
- g) vertices: (1, 2), (1, -4), foci $\left(1, -1 + \frac{\sqrt{65}}{3}\right)$, $\left(1, -1 - \frac{\sqrt{65}}{3}\right)$
- h) vertices: (2, 0), (-2, 0), foci (1, 0), (-1, 0)

vertices: $(3, 6), (3, 2)$, foci $(3, 4 + \sqrt{3}), (3, 4 - \sqrt{3})$

vertices $(-4, 1), (2, 1)$, foci $(-1 - \sqrt{8}, 1), (-1 + \sqrt{8}, 1)$

vertices $(-1, 9), (-1, 1)$, foci $(-1, 5 + 2\sqrt{3}), (-1, 5 - 2\sqrt{3})$

$$31. a) \frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$$

$$f) \frac{(x+2)^2}{16} + \frac{(y-4)^2}{7} = 1$$

$$b) \frac{1}{169}x^2 + \frac{1}{144}y^2 = 1$$

$$g) \frac{24(x-2)^2}{25} + \frac{(y-4)^2}{25} = 1$$

$$c) \frac{1}{3}x^2 + \frac{1}{2}y^2 = 1$$

$$h) \frac{x^2}{25} + \frac{y^2}{21} = 1$$

$$i) \frac{x^2}{36} + \frac{y^2}{16} = 1$$

$$d) \frac{x^2}{20} + \frac{y^2}{36} = 1$$

$$e) \frac{x^2}{36} + \frac{8y^2}{81} = 1 \quad f) \frac{(x-1)^2}{25} + \frac{(y-3)^2}{21} = 1$$

$$36. a) \frac{1}{4}x^2 - \frac{1}{5}y^2 = 1$$

$$f) (y-2)^2 - 4\left(x - \frac{1}{2}\right)^2 = 3$$

$$b) 4x^2 - y^2 = 4$$

$$g) 4y^2 - x^2 = 64$$

$$c) 9x^2 - 4y^2 = 64$$

$$h) \frac{41}{144}(x-4)^2 - \frac{41}{225}(y-2)^2 = 1$$

$$d) y^2 - \frac{9}{4}x^2 = 36$$

$$i) \frac{(y-2)^2}{3} - \frac{(x-7)^2}{12} = 1$$

$$e) \frac{1}{16}(x-6)^2 - \frac{1}{9}(y-4)^2 = 1$$

$$27. a) (y-7)^2 - \frac{(x-1)^2}{3} = 1$$

$$28. a) 3x^2 - y^2 = 3$$

$$b) \frac{x^2}{4} - \frac{y^2}{12} = 1$$

$$b) \frac{y^2}{9} - \frac{x^2}{7} = 1$$

$$c) \frac{7(x-4)^2}{36} - \frac{(y-4)^2}{4} = 1 \quad \text{and} \quad \frac{7}{36}(x-4)^2 - \frac{(x-4)^2}{4} = 1$$

29. a) center $(0, 0)$, vertices $(\pm 4, 0)$, foci $(\pm \sqrt{41}, 0)$, asymptotes $y = \pm \frac{5x}{4}$

b) center $(0, 0)$, vertices $(0, \pm 2)$, foci $(0, \pm \sqrt{29})$, asymptotes $y = \pm \frac{2x}{5}$

c) center $(0, 0)$, vertices $\left(\pm \frac{3}{2}, 0\right)$, foci $\left(\pm \frac{\sqrt{73}}{2}, 0\right)$, asymptote:

$$y = \pm \frac{8x}{3}$$

d) center $(3, -4)$, vertices $(7, -4)$, $(-1, 4)$, foci $(8, -4)$, $(-2, -4)$, asymptote,

$$y + 4 = \pm \frac{3(x - 3)}{4}$$

e) center $(1, -2)$ foci $(1, -2 \pm 2\sqrt{5})$ vertices $(1, 0)$, $(1, -4)$ asymptotes:

$$y + 2 = \pm \frac{(x - 1)}{2}$$

f) center: $(-2, 0)$

vertices: $(1, 0)$, $(-5, 0)$

foci $(-2 \pm \sqrt{34}, 0)$

asymptote: $y = \pm \frac{5(x + 2)}{3}$

g) center: $(1, -1)$

vertices $\left(\frac{7}{3}, -1\right)$, $\left(-\frac{1}{3}, -1\right)$

h) center $(0, 0)$

vertices $(0, \pm 3)$

foci $\left(1 \pm \frac{\sqrt{97}}{3}, -3\right)$

foci $(\pm 3\sqrt{2}, 0)$

asymptote $y + 1 = \frac{\pm 9(x - 1)}{4}$ asymptote: $y = \frac{\pm 3x}{4}$

i) center: $(3, 4)$

vertices: $(3, 6)$, $(3, 2)$

h) center $(0, 0)$

vertices $(-2, 2)$, $(-2, -4)$

foci $(3, 4 \pm 2\sqrt{2})$

foci $(-2, -1 \pm \sqrt{13})$

asymptote $y - 1 = \pm(x - 3)$

asymptote: $y + 2 = \frac{\pm 3(x + 2)}{2}$

Solution for Unit Four

1. a) T

b) T

c) T

d) F

e) T

2. a) All students study their lesson.

b) $3 + 6 > 7$ and $4 - 1 \neq 5$

c) $5 < 3$ and $-3 \geq -5$

d) $3 < 1$ if and only if $3 \neq 1$

e) It is false that you will study or else you will pass your examination.

f) you will not study if and only if you will pass your examination.

g) some Ethiopians are not honest.

3. a) Inverse: If a number is not prime then it is not odd

Converse: If a number is odd then it is prime.

Contra positive: If a number is not odd, then it is not prim.

Inverse: If $5 > 3$ then $-3 > -5$

Converse: If $-3 < -5$, then $5 < 3$

Contra positive: If $-3 > -5$, then $5 > 3$

Converse: If poverty is eradicated, then every Ethiopia does not work hard.

Contra positive: If poverty is not eradicated then every Ethiopian does not work hard.

- a) neither b) tautology c) tautology d) tautology
 e) neither f) tautology g) tautology h) tautology
 i) valid b) invalid c) valid d) valid e) valid f) valid

Solution for Unit Five

The mean is 4.7, the modes are 1 and 4 and median is 4

The mean is 1.05, the mode is 0 and the median is 0

$$\frac{1+2+x+5+y+8}{6} = 4 \Rightarrow 16+x+y=24 \Rightarrow x+y=8. \text{ Thus if the mode is 5, then } x=5 \text{ and } y=3 \text{ and } y=5$$

$x=5, 4$

$$a) \frac{a+d+b+d+c+d+2d}{4} = \frac{a+b+c+d+4d}{4} = \frac{a+b+c+d}{4} + d = \frac{a}{4} + \frac{b}{4} + \frac{c}{4} + \frac{d}{4} + d$$

$$b) \frac{ac+bc+c^2+dc}{4} = \frac{c(a+b+c+d)}{4} = kc$$

Solution from the table reconstruct the c.f table with class interval value

Marks	No. of student (f)	c.f
0-10	15	15
10-20	20	35
20-30	25	60
30-40	24	84
40-50	10	94
50-60	33	127
60-70	71	198
70-80	51	249

Here $N = 249$

b) Calculating the median, $\frac{N}{2} = 124.5 \therefore$ median class is 50 - 60, $L_{mc} = 94$,
 $h = 10$, $f = 33$, $c = 94$

c) Calculating lower quartile Q_1 , $\frac{N}{4} = 62.25$

$\therefore$ lower quartile class is 30 - 40, $L = 30$, $h = 10$, $f = 24$, $c = 60$

$$\therefore Q_1 = \ell + \frac{h}{f} \left(\frac{N}{4} - c \right) = 30 + \frac{10}{24} (62.25 - 60) = 30.94 \text{ marks}$$

iii) Calculation of upper quartile Q_3 , $\frac{3N}{4} = \frac{747}{4} = 186.75$

$\therefore$ upper quartile class is 60 – 40, $\ell = 60$, $h = 10$, $f = 71$, $c = 127$

$$\therefore Q_3 = \ell + \frac{h}{f} \left(\frac{3N}{4} - c \right) = 60 + \frac{10}{71} (186.75 - 127) = 68.41 \text{ marks}$$

7. 13 8. mean = 9.87, median = 9.97

9. $P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5! = 120$

10. $P(10, 4) = 5040$

11. $P(6, 3) = \frac{6!}{3!} = 120$

12. $P(4, 1) + P(4, 2) + P(4, 3) + P(4, 4) = 4 + 12 + 24 + 24 = 64$

13. $P(6, 4) = \frac{6!}{2!} = 6 \times 5 \times 4 \times 3 = 360$

14. Total number of sample point is $P(30, 2) = \frac{30!}{28!} = 870$

15. Total number of possible schedules is $P(5, 3) = \frac{5!}{2!} = 60$

16. Total number of distinct arrangements is $\frac{9!}{3!4!2!} = 1260$

17. a) Total number of possible partitions would be $\frac{8!}{3!3!2!} = 1260$

18. $n = 6$

19. a) The number of ways selecting 3 person from total of 9 is

$$C(9, 3) = \frac{9!}{3!6!} = 84$$

b) $C(4, 2) \times C(5, 1) = 30$ way

c) The number of ways of selecting 2 men where certain man is selected will leave us to select only 1 man from a total of $(4-1) = 3$ men and this is done is

$C(3, 1) = 3$ way and number of ways of selecting a woman from 5 women is $C(5, 1) = 5$ way

$$\therefore C(3, 1) \times C(5, 1) = 15 \text{ way}$$

120 Signals may be made by raising the flags 1, 2, 3, 4 and 5 at a time. Hence the no. of all numbers of signals is $P(5, 1) + P(5, 2) + P(5, 3) + P(5, 4) + P(5, 5)$
 $= 325$ signals

There are $3 \cdot 4 = 12$ books on which 3 are a like 3 others a like, etc.

$$\text{Number of arrangements} = \frac{(3 \cdot 4)!}{3!3!3!} = \frac{12!}{(3!)^4} = 369,600$$

$$C(8, 2) + C(8, 3) + C(8, 4) + \dots + C(8, 8) = 247 \text{ way}$$

$$k = \pm 1$$

$$i) 4^{\text{th}} \text{ term}$$

$$ii) 6^{\text{th}} \text{ term}$$

$$iii) 6^{\text{th}} \text{ term}$$

$$C(9, 3) a^3 b^6$$

$$\frac{9}{7}$$

$$k = \frac{9}{7}$$

$$C(2n+1, r) = C(2n+1, r+1) \Rightarrow r + r + 1 = 2n + 1 \Rightarrow r = n$$

$$r = 6$$

$$\left(\frac{1}{2}\right)^5 (C(5, 2) + C(5, 3) + C(5, 4)) \quad 32. \left(\frac{1}{6}\right)$$

$$\frac{7}{36}$$

$$34. a) \frac{1}{3}$$

$$b) \frac{3}{5}$$

$$c) \frac{11}{15}$$

$$d) \frac{2}{5}$$

$$e) \frac{4}{5}$$

8. Three 6's and two non 6's can occur.

$$C\left(\frac{5}{3}\right) \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2$$

$$36. a) \frac{2}{5}$$

$$b) \frac{1}{5}$$

$$c) \frac{4}{5}$$

$$d) \frac{13}{15}$$

Answer to Review Exercise for Unit - 6

$$a) x = 1, y = -2, z = -\frac{3}{2} \quad b) x = 2, y = -1, z = -4$$

$$c) x = -2, y = 2 \quad d) x = 3, y = \frac{1}{2}, z = -\frac{1}{5}$$

$$x = 1, y = 3 \quad 4. a) x = -\frac{1}{2}, y = 3 \quad b) x = \frac{1}{3}, y = -8$$

$$a. 16$$

$$b.$$

$$a. -10$$

$$b.$$

$$a. \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$$

$$b.$$

$$5$$

$$c.$$

$$4$$

$$12$$

$$c.$$

$$-5$$

$$b. \begin{pmatrix} -7 & -9 \\ -3 & -4 \end{pmatrix}$$

$$c. \begin{pmatrix} -1 & 2 \\ -5 & 9 \end{pmatrix}$$

8. $2x + 4y = 1$ and $x + 3y = 0 \Rightarrow x = \frac{3}{2}$ and $y = -\frac{1}{2}$
9. $33 + 0 + 16m = 1 \Rightarrow m = -2$ and $32 + -2 - 6n = 0 \Rightarrow n = -5$
10. $x = A^{-1}B = \begin{bmatrix} 1 & -5 \\ -2 & 11 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -6 & -1 \\ 14 & 0 \end{bmatrix}$
11. $(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -3 & 7 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix}$
12. a) $x = -2, y = 3$ b) $x = -2, y = 5$
13. $x = \frac{D_x}{D} = \frac{9(4-1)}{6} = \frac{9}{2}, y = \frac{D_y}{D} = \frac{9}{6} = \frac{3}{2}$ and $z = \frac{D_z}{D} = \frac{-36}{6} = -6$
14. $\det(A) = 4 \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 0 + 0 = 4(8-3) = 20$ but $\det(A^T) = \det(A)$ and $\det(A^T A) = \det(A^T) \det(A) \Rightarrow \det(A^T) \det(A) = 20 \times 20 = 400$
15. $-3R_1 + R_3$ denote that -3 times the first row added to the third row. $-3R_1 + R_3$ replace either row 1 or row 3.
16. a) $x = -1, y = -2, z = 3$ b) $x = 2, y = -1, z = -2$
17. a) $\{(-2k-2, k, k) | k \in \mathbb{R}\}$ b) $\{(5k+2, -3k+3, k) | k \in \mathbb{R}\}$

Solution for Unit Seven

1. $\vec{a} + 3\vec{b}, 2\vec{a} + 2\vec{b}, 2\vec{b}, 3\vec{b} - \vec{a}$
2. left for the reader
3. Left for the reader
4. $\sqrt{179}$ unit
5. a) $\vec{AB}$ b) $\vec{AC}$ c) $\vec{AB}$ d) $\vec{AC}$ e) 0 f) $\vec{BD}$ 6. exercise left for
7. $7\frac{2}{7}\vec{i} + \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}$ 8. $\frac{-2}{\sqrt{21}}\vec{i} + \frac{1}{\sqrt{21}}\vec{j} + \frac{4}{\sqrt{21}}\vec{k}$
9. a) 90° b) 45° c) 135° 10. $t = -1$
11. a) 1, b) -11 12. a) $k = -1$, b) $k = -\frac{1}{2}$
13. a) $5\sqrt{2}$ b) $\theta = \cos^{-1}\left(\frac{1}{\sqrt{7}}\right)$ 15. a) 90° b) 300°
16. $\frac{-2}{\sqrt{30}}, \frac{-5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}$
17. a) $\vec{r} = (-3, 4) + t(2, 1)$ b) $\vec{r} = (5, 1) + t(3, -2)$
- $\frac{x+3}{2} = y-4$ $\frac{x-5}{3} = \frac{y-1}{-2}$

$$a) 2\sqrt{3} \quad b) \sqrt{14} + \sqrt{2} \quad c) 2\sqrt{37} \quad d) \frac{1}{\sqrt{6}}i + \frac{1}{\sqrt{6}}j - \frac{2}{\sqrt{6}}k \quad f) 1$$

$$a) \frac{3\sqrt{2}}{2}i + \frac{3\sqrt{2}}{2}j \quad b) 2j \quad c) \frac{-5}{2}i + \frac{5\sqrt{3}}{2}j \quad d) -i + 0j$$

$$\vec{r} = \left(\frac{-2}{3}, 1 \right)$$

$$21. a) x = 3 + 2t, y = -2 + 3t, b) x = 5 - 3t, y = -2 + 6t, z = 1 + t$$

$$a) y = -2x + 8, b) y - x = -1$$

$$a) (x-3)^2 + (y-4)^2 = 4,$$

$$b) (x+1)^2 + (y-1)^2 = 3$$

$$x = 3 + 4t, y = -4 + 3t$$

$$a) (6, -3) b) (7, -3) c) (x-5)^2 + (y+2)^2 = 5$$

$$a) y = -2x - 3, b) (-3, 3) c) y = 2 \quad d) x = -3$$

$$a) (4, -2) b) (x-4)^2 + (y+2)^2 = 6, c) x = 4$$

$$a) (4\sqrt{2}, 2\sqrt{2}), (-3\sqrt{3} - 1, 3 + \sqrt{3}), (3 - \sqrt{3}, -3\sqrt{3} - 1)$$

$$b) \left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \right), \left(-\sqrt{3} - \frac{1}{2}, 1 - \frac{\sqrt{3}}{2} \right), \left(1 + \frac{\sqrt{3}}{3}, \sqrt{3} + \frac{1}{2} \right)$$

$$c) y = x + \sqrt{2}, y + \frac{\sqrt{3}}{2} = \frac{-1}{\sqrt{3}} \left(x + \frac{1}{2} \right), y - \frac{1}{2} = -\sqrt{3} \left(x - \frac{\sqrt{3}}{2} \right)$$

d) left for reader

$$e) x^2 + y^2 = 2, x^2 + y^2 = 2, x^2 + y^2 = 2$$

$$a) (-3, -4) \text{ about origin } (-45) \text{ about } (1, 2)$$

$$b) y = -\frac{1}{2}x - \frac{1}{2} \text{ about origin, } y = -\frac{1}{2}x + \frac{3}{2} \text{ about } (1, 1)$$

$$c) (x+3)^2 + (y+4)^2 = 4, \text{ about origin } (0, 0)$$

$$(x+4)^2 + (y-5)^2 = 4 \text{ about } (1, 2)$$

Solution for Unit Eight

$$a) 6i \quad b) (2\sqrt{3} + 2\sqrt{2})i \quad c) -1 + \sqrt{3}i \quad d) 4\sqrt{2} - 4\sqrt{2}i$$

$$a) (2, -3)$$

$$b) (\sqrt{2} + 1, \sqrt{3} - 4) \quad c) (0, 4) \quad d) (5, 0)$$

$$a) 30 - 7i$$

$$b) (\sqrt{6} + 1 + (\sqrt{3} - \sqrt{3})i) \quad c) 97 - i \quad d) 10 + 6i$$

$$a) x = \frac{2}{13}, y = \frac{23}{13}$$

$$b) x = \frac{47}{65}, y = \frac{14}{65}$$

$$c) x = \frac{47}{65}, y = \frac{1}{65}$$

$$d) x = \frac{31}{26}, y = \frac{-20}{13}$$

5. a) $\overline{Z_1} + \overline{Z_2} = 5, \overline{Z_1} + \overline{Z_2} = 5, \overline{Z_1} \overline{Z_2} = 7 + i, \overline{Z_1} \cdot \overline{Z_2} = 7 + i$
 b) $\overline{Z_1} + \overline{Z_2} = -1 + i, \overline{Z_1} + \overline{Z_2} = -1 + i, \overline{Z_1} \overline{Z_2} = 2 + i, \overline{Z_1} \cdot \overline{Z_2} = 2 + i$
 c) $\overline{Z_1} + \overline{Z_2} = 2 - \sqrt{3}i, \overline{Z_1} + \overline{Z_2} = 2 - \sqrt{3}i, \overline{Z_1} \overline{Z_2} = -2\sqrt{3}i = \overline{Z_1} \cdot \overline{Z_2} = -2\sqrt{3}i$
 d) $\overline{Z_1} + \overline{Z_2} = 3 + (2 - \sqrt{5})i, \overline{Z_1} + \overline{Z_2} = 3 + (2 - \sqrt{5})i, \overline{Z_1} \overline{Z_2} = -2\sqrt{5} - 3\sqrt{5}i, \overline{Z_1} \cdot \overline{Z_2} = 2\sqrt{5} - 3\sqrt{5}i$
- 6) a) $\frac{23}{65} + \frac{184i}{65}$ b) $\frac{11}{10} + \frac{4i}{5}$ c) $-i$ 7) a) 10 b) 1
- 8) a) $\cos 300^\circ + i \sin 300^\circ$ b) $\sqrt{17} [\cos \theta + i \sin \theta]$ so that $\tan \theta = -$
 d) $\frac{1}{\sqrt{2}} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$ 9) a) $-32i$ b) $1024 - 1024i$ c) $-i$
- 10) a) $2(\cos 60^\circ + i \sin 60^\circ), 2(\cos 180^\circ + i \sin 180^\circ), 2 [\cos 300^\circ + i \sin 300^\circ]$
 $\sqrt[3]{2} [\cos 80^\circ + i \sin 80^\circ]$
 b) $\sqrt[3]{2} [\cos 200^\circ + i \sin 200^\circ]$
 $\sqrt[3]{2} [\cos 320^\circ + i \sin 320^\circ]$
- 11) $2\sqrt{2} + 2\sqrt{6}i, -2\sqrt{2} - 2\sqrt{6}i$
- 12) a) $x = \frac{5}{13}, y = \frac{14}{13}$ b) $x = -4, y = 6$ c) $x = 3, y = -1$
- 13) a) center $(-1, 1)$, radius $r = \sqrt{2}$ b) center $(2, -3)$, radius $r = 2$
- 14) $y = \cos \theta \pm i \sin \theta$

Solution for Unit Nine

1. a) $\frac{\pi}{2}$ b) $\frac{5\pi}{4}, \frac{7\pi}{4}$ c) $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$
2. a) $2, \frac{\pi}{2}, 2\pi$ b) $1, \frac{\pi}{8}, \pi$ c) $4, \frac{-\pi}{4}, 3\pi$ d) $\frac{5}{4}, \frac{2\pi}{3}, \frac{2\pi}{3}$
3. a) $\frac{\pi}{8}, \frac{\pi}{2}$ b) $-3\pi, 6\pi$ 5. a) $y = 3 \cos \left(\frac{3}{2}x + \frac{3\pi}{8} \right)$
- b) $y = \tan \left(\frac{x}{2} - \frac{\pi}{4} \right)$ c) $y = \sec \left(\frac{x}{2} - \frac{3\pi}{8} \right)$ 6. a) $\frac{5\pi}{6}$ b) $\frac{-\pi}{4}$
- c) $\frac{\pi}{3}$ d) $\frac{1}{2}$ e) 0 f) $\frac{24}{25}$ g) $\frac{-\pi}{6}$ h) $\frac{1 - \sqrt{1 - x^2}}{x}$

$$b) 2x\sqrt{1-x^2} \quad c) \sqrt{1-10x^2+9x^4}-3x^2$$

$$d) \frac{1}{\sqrt{1-x^2}}$$

$$e) \frac{2+\sqrt{15}}{6}$$

$$f) \frac{1}{6}$$

$$g) y = \frac{1}{3} \tan 5x$$

$$h) y = \cos\left(x - \frac{\pi}{3}\right) + 3$$

$$b) \frac{1}{5}(3\sqrt{7}-4\sqrt{3}) \quad 9. \text{ Exercise left for reader}$$

$$a) \frac{13}{5}$$

$$b) \frac{3}{2}$$

$$c) \frac{4}{5}$$

(d, e) exercise left for reader.

$$a) \frac{435}{533}$$

$$b) \frac{92}{525}$$

$$c) \frac{308}{533}$$

d) quadrant I

Solution for unit ten

$$a. A_1 = -4$$

$$b. A_{20} = A_1 + 19d = -4 + 57 = 53$$

$$c. A_n = 3n - 7$$

$$d. A_{n+1} = 3n - 4$$

$$a. d = \frac{A_8 - A_4}{8-4} = \frac{13-11}{4} = \frac{1}{2}$$

$$b. A_1 = \frac{19}{2}$$

$$c. A_n = \frac{1}{2}n + 9$$

$$\text{Since } d = \frac{A_k - A_m}{k-m} = \frac{2m+1-2k-1}{k-m} = -2 \Rightarrow A_{k+m} = 1$$

$$\{102, 108, \dots, 996\} \Rightarrow A_n = A_1 + (n-1)6 \Rightarrow 996 = 102 + 6n - 6 \Rightarrow n = 150$$

$$a. d = 3$$

$$b. d = \frac{-7}{2}$$

$$a. d = \frac{a_{307} - a_{309}}{307-309} \Rightarrow a_{307} - a_{309} = 6$$

$$b. d = \frac{a_{200} - a_{204}}{200-204} = -3 \Rightarrow a_{200} - a_{204} = 12$$

$$7. a. d = \frac{7-3}{n+1} = \frac{4}{n+1}$$

$$b. d = \frac{3-2}{n+1} = \frac{1}{n+1} \quad c. d = \frac{k-m}{n+1}$$

$$8. a. G_6 = \frac{1}{4}$$

$$b. G_8 = (8) \left(\frac{1}{27} \right) = \frac{1}{16}$$

$$c. G_n = 8 \left(\frac{1}{2} \right)^{n-1} = 16 \left(\frac{1}{2} \right)^n$$

$$9. \quad a. \quad r = 3 \quad b. \quad G_1 = \frac{1}{9} \quad c. \quad G_n = G_1 r^{n-1}$$

$$10. \quad a. \quad r = 4 \quad b. \quad G_1 = \frac{1}{16} \quad c. \quad G_n = 4^{n-1}$$

$$11. \quad r = -\frac{1}{2}, G_1 = \sqrt{2}, G_5 = \frac{1}{\sqrt{128}} \text{ and } G_n = \sqrt{2} \left(-\frac{1}{2} \right)^{n-1}$$

$$12. \quad \frac{9}{x} = \frac{y}{9} \Rightarrow xy = 81 \text{ and } 15 - y = 15 \Rightarrow y = 30 \Rightarrow x = 27, r = 3$$

$$13. \quad \frac{-5x+4}{x+1} = \frac{2x+8}{-5x+4} \Rightarrow (-5x+4)^2 = (x+1)(2x+8) \Rightarrow x = 2$$

$$14. \quad a. \quad 15 \times 6 = 90 \quad b. (21-4+1)(9) = 162 \quad c. (100-12+1)(7) = 623$$

$$d. \quad 110 \quad e. \quad 177$$

$$f. \quad S_n = 2 - \frac{2}{n+1} \Rightarrow S_{13} = 2 - \frac{2}{14} = \frac{26}{14} = \frac{13}{7}$$

$$15. \quad a. \quad S_{10} = \frac{10}{2} [A_1 + A_{10}] = 5(8 + 35) = 215$$

$$b. \quad S_{20} = \frac{20}{2} [A_1 + A_{20}] = 730$$

$$c. \quad \sum_{n=1}^{24} (5 + 3n) = 12(8 + 77) = 1020$$

$$d. \quad \sum_{n=1}^{12} (5 + 3n) = 6(8 + 4) = 294$$

$$16. \quad a. \quad A_8 = 19, A_4 = 11 \text{ and } S_{10} = S[2A_1 + 9d] = S(28) = 140$$

$$b. \quad A_1 = 5, A_{10} = 23 \text{ and } A_n = 2n + 3$$

$$17. \quad a. \quad d = 2 \quad b. \quad A_n = 2n + 5 \quad c. \quad \sum_{n=1}^n 2n + 5 = n^2 + 6n$$

$$d. \quad S_{20} = 520$$

- a. $S_n = \frac{n}{2}[4 + 3n + 1] = \frac{1}{2}(3n^2 + 5n)$ b. $S_n = n^2 + 4n$
- c. $S_n = \frac{1}{2}(5n^2 + n)$
- a. $A_n = S_n - S_{n-1} = 6n - 1$ b. $A_n = S_n - S_{n-1} = 5n - 2$
- a. $S_{20} = 10[A_1 + A_{20}] = 10[2 + 59] = 610$
- b. $S_{30} = \frac{30}{2}[A_1 + a_{30}] = 750$
- a. $a_4 + a_5 + \dots + a_{29} \Rightarrow S_{26} = 13[2a_4 + 25(2)] = 13(18 + 52) = 910$
- a. $S_{25} = \frac{25}{2}[104 + 296] = 5,000$ b. $S_{28} = \frac{28}{2}[105 + 294] = 5,516$
- 1830 24. $d = 4$, and $A_n = 4n - 5$
- a. $S_{60} = 12,780$ b. $S_{15} = \frac{15}{2}[-395 + -53] = -3,120$
- a. $S_{20} = 530$ b. $S_{19} = \frac{19}{2}[23 + 59] = 779$ c. $S_{15} = 675$
- d. $S_{99} = \log 100 = 10$ e. $S_{100} = \frac{25}{51}$ f. $S_{70} = 70x^3$
- a. $S_{79} = 79 \times 979$ b. $S_{69} = \frac{69}{2}[104 + 988]$ c. $S_{20} = 20,424$
- a. 25 b. 15 c. $\frac{5n}{2}$ d. $\frac{n}{2}(m + k)$
- a. $S_{15} = 8(2(8) + 30) = 368$
- a. $S_{39} = 4017$ b. $S_{40} = 7980$
- a. $S_6 = 195$ b. $S_8 = 381$ c. $S_{10} = 3069$
- d. $3[2^n - 1]$
- a. $S_7 = 381$ b. $S_7 = \frac{16}{3}[1 - 3^{-7}]$ c. $3\left[\left(\frac{3}{2}\right)^{10} - 1\right]$
- a. $r = 2$ b. $S_6 = \frac{65}{2}$ c. $S_n = \frac{1}{2}(2^n - 1)$

34. a. $\frac{2}{3}$ b. 3 c. $\frac{1}{7}$ d. $\frac{3}{2}$
35. a. $\frac{10}{3}$ b. $\frac{49}{30}$ c. $\frac{5}{8}$ d. 200
- e. $\frac{40}{3}$ f. $\frac{15}{4}$ g. $\frac{125}{4}$ h. $\frac{4}{3}$
- i. 32 j. 108 k. $\frac{35}{44}$ l. 5
36. a. $3^{1+\frac{1}{3}+\frac{1}{9}+\dots} = 3^{1+\frac{1}{3}} = 3\sqrt{3}$ b. $4^{\frac{1}{1-\frac{1}{5}}} = \sqrt[4]{1024}$
37. $k = \frac{2}{3}$
38. a. $k = -2$ b. $k = -3$ c. $k = -2$

Solution for unit eleven

1. a. i) $\left[\frac{5}{4}, \infty\right)$ ii. $\text{lub} = \frac{5}{4}$ iii. $\left(-\infty, -\frac{2}{3}\right)$ iv. $\text{glb} = -\frac{2}{3}$
- b. i) $(1, \infty)$ ii. $\text{lub} = 1$ iii. $\left(-\infty, -\frac{1}{4}\right)$ iv. $\text{glb} = -\frac{1}{4}$
- c. i) $\left[\frac{5}{3}, \infty\right)$ ii. $\text{lub} = \frac{5}{3}$ iii. $(-\infty, -4)$ iv. $\text{glb} = -4$
- d. i) $[e^{-2}, \infty)$ ii. $\text{lub} = e^{-2}$ iii. $(-\infty, 0)$ iv. $\text{glb} = 0$
- e. i) $\left[\frac{32}{3}, \infty\right)$ ii. $\text{lub} = \frac{32}{3}$ iii. $(-\infty, 0)$ iv. $\text{glb} = 0$
- f. i) $\left[\frac{2}{3}, \infty\right)$ ii. $\text{lub} = \frac{2}{3}$ iii. $(-\infty, 0]$ iv. $\text{glb} = 0$
2. a. increasing b. st. decreasing c. st. increasing
- d. st. decreasing e. st. decreasing f. increasing
- g. decreasing h. neither i. st. increasing
- j. decreasing k. increasing
3. a. convergent, limit value = 1
- b. divergent, limit value doesn't exist (2 or -2)

3. c. convergent, limit value $\frac{1}{2}$
 d. convergent, limit value e^2
 e. divergent, doesn't exist

Limit value: a. $\frac{25}{9}$ b. $\frac{-11}{9}$

a. $\frac{2}{3}$ b. $-\frac{1}{3}$ c. $\frac{-\sqrt{3}}{2}$ d. $-\frac{1}{3}$

a. $\frac{1}{2}$ b. $-\frac{3}{10}$

a. $\frac{10}{7}$ b. $\frac{4}{3}$ c. $\frac{1}{25}$ d. $\frac{1}{3\sqrt[3]{a^2}}$

e. nt^{n-1} f. $-\frac{1}{9}$ g. $\frac{1}{6}$ h. 3

i. $\frac{1}{3\sqrt[3]{25}}$ j. 6 k. 1 l. $\frac{1}{\sqrt{6}}$

a. e^{-1} b. $e^{\frac{20}{3}}$ c. e^5 d. e^{-4}

a. $k = \frac{-\ln 5}{2}$ b. $k = \frac{-2}{3} \ln 4$

a. $\frac{1}{4}$ b. $\frac{1}{5}$ c. -1 d. $\frac{1}{\pi}$

e. -2 f. $\frac{1}{1+\pi}$ g. $-\frac{7}{2}$ h. 8

a. 4 b. $-\frac{1}{\sqrt{3}}$ c. 8

a. $-\infty$ b. $-\infty$ c. $-\infty$ d. $\pm\infty$
 e. ∞ f. ∞ g. $-\infty$ h. $-\infty$ i. $-\infty$

a. $(-\infty, -2]$ and $[2, \infty)$ b. $\{x: -3 \leq x \leq 3\}$ c. $\{x: x \neq 3, -3\}$

d. $\{x \in \mathbb{R}: x \neq 3\}$ e. $\{x \in \mathbb{R}: x > 2\}$

f. for all integer k , $[k, k+1]$ is continuous

g. $(-\infty, 0) \cup [2, \infty)$ h. $(-\infty, \ln 2) \cup [3, \infty)$ i. $(0, \infty)$

j. $(-\infty, 2]$ and $(2, \infty)$ k. at each $x \in (-\infty, \infty)$

14. a. $c = \frac{-3}{2}$ b. $a = \frac{5}{4}$ c. $a = e^2$ d. $c = \pm \frac{1}{2}$

Solution for unit - 12

1. a. $y = 2x - 2\pi + 1$ b. $y = -6x + 3$ c. $y = 3x - 3$

d. $y = 10x - 13$ e. $\frac{1}{2}x - \frac{1}{2}$ f. $y = \frac{3}{2}x$

g. $y = e^{-\sqrt{2}}x$ h. $y - \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$

i. $y - 1 = \frac{x - 8}{10 \ln 10}$ j. $y = 4x - 2$ k. $y = 3x + 2$

l. $y = 3x + 1$ m. $y = e(4x + 3)$ n. $y = -x$

2. a. $f'(x) = 0 \Rightarrow 6x^2 - 6 = 0 \Rightarrow x = \pm 1$

a. $f''(x) > 0 \Rightarrow 6x^2 - 6 > 0 \Rightarrow x \in (-\infty, -1) \cup (1, \infty)$

b. $f''(x) < 0 \Rightarrow 6x^2 - 6 < 0 \Rightarrow x \in (-1, 1)$

c. $f'(x) = 2 \Rightarrow 6x^2 - 6 = 2 \Rightarrow x = \pm \frac{2\sqrt{3}}{3}$

d. $f'(x) = -2 \Rightarrow 6x^2 - 6 = -2 \Rightarrow x = \pm \sqrt{\frac{2}{3}}$

3. a. $(0, 2) \Rightarrow y - f(0) = f'(0)(x - 0) \Rightarrow y - 2 = -15x \Rightarrow y = -15x + 2$

$(3, 2) \Rightarrow y - f(3) = f'(3)(x - 3) \Rightarrow y - 2 = 24(x - 3)$

b. $(0, 2) \Rightarrow y - 2 = -15x$

4. a. $(-\infty, \infty)$ b. $x \in [-5, \infty)$ c. $x \in \left[-\frac{5}{3}, \frac{5}{3}\right]$ d. $(-\infty, 3]$ and $[3, \infty)$

5. a. $\frac{-\ln 3}{x(\ln x)^2}$ b. $\frac{5}{2x \ln 4}$ c. $\frac{2x}{(\ln 2)(x^2 + 5)}$

d. $\frac{x}{(x^2 + 1) \ln 10}$ e. $\frac{6}{6x + 5} - \frac{6x + 1}{3x^2 + x + 1}$

f. $\frac{-\sin x}{\sqrt{3x - 1}} - \frac{3}{2\sqrt{(3x - 1)^2}}$ g. $2x \cos(x^2 + 1)$

h. $\frac{6x^2}{\sqrt{1 + x^4}} - \frac{8x^6}{2(\sqrt{1 + x^4})^3}$ i. $\frac{2x + 5}{2\sqrt{x^2 + 5x + 6}}$

- $\frac{x+2}{\sqrt{x^2+4x}} e^{\sqrt{x^2+4x}}$
 1. $2 \sec^2(x+3) \tan(x+3)$
 2. $3^{x^2} \ln 3(-2x)$
 3. $\frac{9(4x-1)^2}{(2-5x)^4}$
 4. a. 7 b. $\frac{47}{16}$
 5. $F'(-1) = -3$
 6. $F'(x) = 3x^2 f(x) + x^3 f'(x) + f'(g(x)) g'(x) \Rightarrow F'(1) = 7e^3 + 4e^{13}$
 7. $f''(x) = \frac{-2x^2 + 8x - 16}{(x^2 - 4x)^2} \Rightarrow f''(-1) = \frac{-2 - 8 - 16}{25} = \frac{-26}{25}$
 8. $G'(-1) = 24$
 9. a. $f''(x) = (-4)^n e^{-4x}$
 10. $f''(x) = e^x (x+n)$
 11. a. $y = -2x - 4$
 12. $f''(1) = -e^{-1}$
 13. $g''(x) = f''(2x+4)(2'')$
 14. a. $f''(x) = \frac{n!}{(3-x)^{n+1}}$
 15. k. $2x(\log_2 x) + \frac{x^2+3}{x \ln 2}$
 16. m. $\frac{-5^x \ln 5}{(5^x + 6)^2}$
 17. o. $\frac{\cos \sqrt{x+4} (e^{\sin \sqrt{x+4}})}{2\sqrt{x+4}}$
 18. c. $\frac{-4\sqrt{5}}{5}$
 19. 8. $f'(3) = \frac{1}{4}$
 20. 12. $G'(1) = \frac{-8}{9}$
 21. b. $f''(0) = (-4)^n$
 22. 15. $g'(1) = \frac{1}{6}(1 - \ln 6)$
 23. b. $5y - x = -42$
 24. 18. $f'''(0) = (\pi \ln 3)^3$
 25. b. $f''(x) = \frac{(-1)^n n!}{(2+x)^{n+1}}$

Solution for unit - 13

1. a. i) 2, -2 ii) max. value 28 iii) min. value -16 iv) [-16, 28]
 b. i) 0, $\frac{1}{8}$ ii) max. value 3 iii) min. value 0 iv) [0, 5]
 c. i) 1 ii) max. value 5 iii) min. value 0 iv) [0, 3]

- d. i) $-\frac{1}{2}$ ii) max. value $\frac{7}{2}$ iii) min. value 0 iv) $\left[0, \frac{7}{2}\right]$
- e. i) $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ ii) max. value $\left(\frac{\pi}{6} + \sqrt{3}\right)$ iii) min. value $\left(\frac{\pi}{6} - \sqrt{3}\right)$
- f. i) $-2, 0, 2$ ii) max. value 27 iii) min. value -48
- g. i) 1, 5 ii) max. value 143 iii) min. value 10 iv) 10, 143
- h. i) 2 ii) max. value 3 iii) min. value 1 iv) $[1, 3]$
2. a) i) $(-\infty, -2]$ and $[2, \infty)$ ii) $-2 \leq x \leq 2$ iii) at $x = -2$ and $x = 2$
- b) i) $[-4, 4]$ ii) $(-\infty, -4]$ and $[4, \infty)$ iii) at $x = -4$ and at $x = 4$
- c) i) $(-\infty, \infty)$ ii) no interval iii) no extreme value
- d) i) no interval ii) decreasing for all x $(-\infty, \infty)$ iii) no local extreme value
- e) i) $(-\infty, -1]$ and $[1, \infty)$ ii) $[-1, 1]$ iii) at $x = -1$, max. at $x = 1$, min
- f) i) $(-\infty, -1]$ and $[1, \infty)$ ii) $[-1, 1]$ iii) at $x = -1$ and at $x = 1$
- g) i) $(-\infty, 3]$ ii) $[3, \infty)$ iii) at $x = 3$
- h) i) no interval value ii) $\mathbb{R} \setminus \{7\}$ iii) no extreme value
- i) i) $[-1, \infty)$ ii) $(-\infty, 1]$ iii) at $x = -1$
3. a) $b = 1, a = 1$ b) $k = -4$ c) $k = \frac{1}{3}$
- d) $k = 9$
4. a) i) $(0, \infty)$ ii) $(-\infty, 0)$ iii) at $(0, f(0))$
- b) i) $(-\infty, \infty)$ ii) No
- iii) no inflection
- c) i) $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$ ii) $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
- iii) at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$
- d) i) $\left(\frac{1}{5}, \infty\right)$ ii) $\left(-\infty, \frac{1}{5}\right)$ iii) at $x = \frac{1}{5}$
- e) i) $(-\infty, -3)$ ii) $(-3, -1)$ iii) at $x = -1$ and $x = -3$
5. a) 6 and 6 b) $\frac{15}{2}$ and $\frac{15}{2}$ c) $2\sqrt{6}$ and $2\sqrt{6}$ d) $\frac{4}{3}, 0, \frac{2}{3}$

a) $\frac{4}{5}m$ and $\frac{4}{5}m$ b) 30 m and 30 m c) $4\sqrt{2}$ and $4\sqrt{2}$ d) 9m and 9m

a) $(-\sqrt{3}, 3)$ and $(\sqrt{3}, 3)$ b. $\left(-\sqrt{\frac{5}{2}}, \frac{3}{2}\right)$ and $\left(\frac{5}{2}, \frac{3}{2}\right)$ c) $\left(\frac{3}{5}, \sqrt{\frac{3}{5}}\right)$

$$h = 2 \text{ cm}, r = 10\sqrt{\frac{15}{2\pi}}$$

$$2\pi r + 4x = 24 \text{ m} \Rightarrow x = 6 - \frac{1}{2}\pi r \text{ and } A = x^2 + \pi r^2$$

$$\Rightarrow A = \left(6 - \frac{1}{2}\pi r\right)^2 + \pi r^2 \Rightarrow A'(r) = 2\pi r + \pi^2 r - 6\pi = A'(r) = 0$$

$$\Rightarrow r = \frac{6}{\pi + 2} \text{ and } x = \frac{3\pi + 12}{\pi + 2}$$

10. a) $4\pi \text{ cm/sec}$ b) $20\pi \text{ cm/sec}$ 11. a) $\frac{1}{64\pi} \text{ cm/sec}$ b) $\frac{1}{2} \text{ cm/sec}$

12. $6\sqrt{2} \text{ cm}$ by $3\sqrt{2}$

13. $\frac{9}{4} \text{ ft. sec}$

$$14. v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{1}{4\pi r^2} \left(\frac{dv}{dt}\right), \text{ then a) } r = 2 \text{ cm}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi(4)} \left(\frac{3 \text{ cm}^3}{\text{min}}\right) = \frac{3}{16\pi} \text{ cm/min} \quad \text{b) } \frac{dr}{dt} = \frac{1}{48\pi} \text{ cm/min}$$

$$15. v = \frac{1}{3}\pi r^2 h \text{ and } \frac{r}{h} = \frac{2}{6} \Rightarrow r = \frac{1}{3}h \Rightarrow v = \frac{1}{3}\pi \frac{h^3}{9}$$

$$v = \frac{\pi h^3}{27} \Rightarrow \frac{dv}{dt} = \frac{3\pi h^2}{27} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{9}{\pi h^2} \left(\frac{dv}{dt}\right) = \frac{9}{\pi(16)} (2) = \frac{9}{8} \text{ cm/min}$$

Solution for unit - 14

1. a. $\sqrt{3}x + c$ b. $\frac{x^{\sqrt{2}+1}}{\sqrt{2}+1} + c$ c. $\frac{-\sqrt{3-4x}}{6} + c$
d. $\frac{1}{3}\sqrt{2x-1} + c$ e. $\frac{(3x-1)^5}{15} + c$ f. $\frac{1}{7}(\sqrt[3]{3x+1}) + c$

2. a. $\frac{e^{10}}{4} + c$ b. $\frac{e^{3x-4}}{3} + c$ c. $\frac{3^{4x}}{4 \ln 3} + c$

d. $\frac{-e^{-2x}}{2} + e^{-x} + c$ e. $\frac{-5}{2}e^{-2x} + e^{-x} + c$ f. $\frac{-2}{3} \frac{24^x}{\ln 24} + c$

3. a. $\ln |x| + c$ b. $\frac{1}{4} \ln |4x-1| + c$ c. $-\frac{1}{5} \ln |2-5x| + c$

d. $\frac{-\sqrt{2}}{7} \ln |1-7x| + c$ e. $e^{-x} - e^{-x} + 2x + c$ f. $2\sqrt{x-1} - \frac{1}{2}e^x$

g. $\int 2\sqrt{e^x} - e^{-x} + c$ h. $-\frac{1}{x} + c$ i. $-\frac{1}{x+3} + c$

j. $-\frac{1}{2}(x-4)^2 + c$ k. $\frac{1}{6-9x} + c$

4. a. $2e^{x+3} + c$ b. $e^{x^4} + c$ c. $\frac{2}{3}(\sqrt{x+3})^3 - 4\sqrt{x+3} + c$

d. $6\sqrt{x^2+4} + c$ e. $\frac{(4x^2+2x+7)}{14} + c$ f. $\frac{1}{2} \ln |1-2x+4x^2| + c$

g. $\frac{(2x+3)^9}{2} + c$ h. $\frac{(4x-3)^{11}}{44} + c$ i. $\frac{2}{3}\sqrt{x^3-2} + c$

5. a. $\frac{(\ln x)^3}{3} + c$ b. $\frac{-1}{2(\ln x)^2} + c$ c. $\ln |\ln x| + c$

6. a. $\frac{1}{12} \ln \left| \frac{2x-3}{2x+3} \right| + c$ b. $\frac{1}{2} \ln |x^2+4x+3| + c$

c. $\frac{3}{2} \ln |2x-1| - 2 \ln |x+1|$ d. $5 \ln |x-2| - \ln |x+2| - 3 \ln |x| + c$

e. $\ln \left| \frac{x^2+1}{x} \right| + c$

f. $\frac{1}{16} \ln \left| \frac{4x^2-1}{4x^2+1} \right| + c$

7. a. $x^2 e^x - 2x e^x - 2x e^x + 2e^x + c$

b. $-e^{-x}(x^3+3x^2+6x+6) + c$

c. $\frac{1}{2} e^{2x} \left(x^2 - x + \frac{1}{2} \right) + c$

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$$d. \frac{1}{2}x^2 \ln|x+1| - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2} \ln|x+1| + c$$

$$e. x \ln|3x| - 3x + c \quad f. x(\ln x)^2 - 2x \ln x - 2x + c$$

$$g. x^2 \sin x + 2x \cos x - 2 \sin x + c$$

$$h. \frac{1}{2}x^3 \sin 2x + \frac{3}{4}x^2 \cos 2x + \frac{3}{4}x \sin 2x + \frac{3}{8} \cos 2x$$

$$8. a. \frac{e^x}{5}(\sin 2x - 2 \cos 2x) + c \quad b. \frac{e^{2x}}{5}(2 \cos x + \sin x) + c$$

$$c. \frac{e^{-x}}{2}(-\sin x - \cos x) + c$$

$$9. a. \frac{9}{2} \quad b. \frac{13}{2} \quad c. \frac{23}{3} \quad d. 0$$

$$e. \frac{54 - 4\sqrt{2}}{9} \quad f. 2 \quad g. \frac{4}{15} \quad h. \frac{3\sqrt{3}}{4} \quad i. \frac{1}{3}$$

$$10. a. \ln 2 \quad b. \frac{-22}{15} \quad c. \ln \sqrt{2} \quad d. e - 1$$

$$e. \frac{-\pi}{2} \quad f. \frac{\pi}{2} - 1 \quad g. \frac{e}{2}[\sin(1) - \cos(1)] + \frac{1}{2}$$

$$h. \frac{e^2(\sin 2 - \cos 2) + 1}{2} \quad i. \frac{24 \ln 2 - 7}{9}$$

$$11. a. \pi \quad b. \ln 2 \quad c. \frac{\pi}{4} \quad d. 2 \log 2 - 1 \quad e. \sqrt{3} - \sqrt{2}$$

$$f. \frac{4}{3} \quad g. 2 \log 2 \quad h. 2e^2 - e \quad i. \ln 4 - \frac{3}{4}$$

$$j. \frac{74}{9} \quad k. \frac{7}{18} \quad l. \frac{2}{3}$$

$$12. a. \frac{125}{6} \quad b. 2\sqrt{2} \quad c. 24 \quad d. 8$$

$$e. \frac{7}{6} \quad f. \frac{64}{3} \quad g. \frac{8}{\ln 9}$$

$$13. a. \frac{125}{6} \quad b. \frac{10}{3} \quad c. \frac{32}{3}$$

14. a. $\frac{20}{3}\pi$ b. $\frac{196}{5}\pi$ c. $\frac{625}{6}\pi$

d. $v = 2\pi \int_0^2 (16x^2 - x^6) dx = 2\pi \left[\frac{16x^3}{3} - \frac{x^7}{7} \right]_0^2 = \frac{1024\pi}{21}$

e. $v = 2\pi \int_0^1 (x^2 - x^4) dx + 2\pi \int_1^3 (x^4 - x^2) dx$
 $= \frac{4\pi}{15} + \frac{1188\pi}{15} = \frac{1192\pi}{15}$

15. a. $\text{vol} = \pi \int_2^6 (\sqrt{y-2})^2 dy = \pi \left[\frac{y^2}{2} - 2y \right]_2^6 = 8\pi$

b. $\text{vol} = \pi \int_{-1}^2 \sqrt{y+1}^2 dy = \pi \left[\frac{y^2}{2} + y \right]_{-1}^2 = \frac{9\pi}{2}$

c. $\text{vol} = (e-2)\pi$

Solution for unit - 15

1. a. 3 b. 1 c. $\sqrt{13}$ d. $\sqrt{5}$ e. $\sqrt{10}$ f. 2

2. (0, 0, 0) and (0, 2, 0) 3. a. $(x-1)^2 + y^2 + (z+2)^2 = 21$

b. $(x-3)^2 + (y+1)^2 + (z-2)^2 = 14$

4. $5 - 3 = 2$

5. a. outside sphere b. on the sphere c. inside the sphere

6. a. no intersection b. $(0, 3 \pm \sqrt{3}, 0)$ c. no intersection

7. a. (0, -4, 0) and (0, 6, 0) b. $(2 \pm 2\sqrt{7}, 0, 0)$

8. $r = 5$ 9. a. $k = \pm 6$

10. $(x-1)^2 + (y+2)^2 + (z+3)^2 = r^2 = -k + 14 \Rightarrow 0 -k + 14 = 5 \Rightarrow k = 9$

